Existence and uniqueness of a global weak solution of a Darcy-Nernst-Planck-Poisson system

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This paper deals with the analytical investigations of a nonstationary Darcy-Nernst-Planck-Poisson-type system which is a mathematical model describing electrolyte solutions. For this nonlinear fully coupled system of partial differential equations we prove existence and uniqueness of global weak solutions. Furthermore, we establish physical properties for the number densities such as nonnegativity and boundedness in $L^\infty((0,T) \times \Omega)$.

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1 Introduction

Stokes-Nernst-Planck-Poisson-type systems (SNPP-systems) are well known pore-scale models describing electrokinetic phenomena in numerous real-world applications including areas of colloid chemistry [1, 2], electro-hydrodynamics [3] and semiconductor devices [4]. Nevertheless, SNPP-systems are still the subject of current research, concerning among other questions the following three topics: Firstly, the development and justification of such models in different situations is not answered satisfactory. Recently, a thermodynamically consistent derivation was carried out, e.g., in [5]. Especially the important drift-diffusion (van Roosbroeck) subsystem is motivated, e.g., in [6] and is derived from a molecular model with methods from statistical physics in [7]. Secondly, many efforts have been undertaken to include new and sophisticated effects such as size exclusion [8], macromolecule dynamics [9], and nanopore blocking [10]. Thirdly, SNPP-systems are as well of interest from an analytical point view due to their nonlinear structure and have been investigated for quite some time, e.g., in [4, 11, 12, 13, 14] and [15].

In the framework of multiscale modeling, nonstationary Darcy-Nernst-Planck-Poisson-type systems (DNPP-systems) are discovered as homogenized "equivalents" of SNPP-systems,
The idea in considering homogenized versions of SNPP-systems is to obtain “equivalent” systems of partial differential equations that reasonably describe the effective macroscopic outcome of competing microscopic processes. Thus, DNPP-systems are of great interest since they offer the possibility to describe field-scale effects that arise from pore-scale electrokinetic phenomena. Several publications concerning the homogenization of SNPP-systems and its modifications are present in the literature. Formal upscaling attempts have been done among others in [12, 17, 18] and [19] and rigorous homogenization results have been published recently in [20, 21] and [16]. Moreover, numerical simulations are found for instance in [22].

For DNPP-systems rigorous analytical results are still lacking although such results are very interesting from an analytical point of view. The contribution of this paper is to provide a rigorous global existence and uniqueness result for a DNPP-system including regularity properties such as boundedness in $L^\infty((0,T) \times \Omega)$. In particular, this rigorous results extend the previous conclusions of [14], where a rather formal proof in a similar situation was given. Further, we pursue the achievements of [15], since we drop the so-called volume additivity constraint and show global existence even in the sense of $L^\infty((0,T) \times \Omega)$. It is interesting to note that nonnegativity in combination with the volume additivity assumption is equivalent to a $L^\infty$-bound. This means, in [15] the boundedness was included by assumption whereas we prove that by means of a $L^\infty$-estimate boundedness is inherently satisfied.

The paper is organized as follows: In Section 2 we present the Darcy-Nernst-Planck-Poisson system. Section 3 is concerned with physical properties and regularity of the solutions, as well as with the existence and uniqueness of global weak solutions of the DNPP-system. In Theorem 3.5, we establish an $L^\infty$-estimate via Moser’s iteration technique, which was introduced in [23] and [24]. A fixed point approach is applied to proof the main results, which are stated in Theorem 3.10 and Theorem 3.11, respectively.

### 2 Model Description

For ease of presentation, we consider only two oppositely charged species, namely positively (+) and negatively (-) charged particles, which are modeled by number densities $c^+$ and $c^-$. In the remainder of the paper, the notation $c^\pm$ (and $c^\mp$) is used in order to treat the equations for $c^+$ and $c^-$ simultaneously. The upper signs correspond to the first equation, the lower signs to the second one.

When these charged particles start to move in a domain $\Omega$ with a charged boundary $\partial \Omega$, they form a free charge density and interact with the surface potential. This leads to spatially inhomogeneous distributions generating an electric field and to an electric body force in the surrounding fluid. Thereby, an electroosmotic flow develops which, on the other hand, feeds back on the particles. This leads to a complicated interplay between electrophoretic movement and electroosmotic flow in a varying electric field. DNPP-systems model such phenomena based on three conservation laws.

- The electric field $E$ satisfies Gauss’s law with the moving particles as free charge density. This couples the electric field to the number densities $c^\pm$. Assuming an irrotational electric field allows to write $E = -\nabla \Phi$, with $\Phi$ being the corresponding electrostatic potential.
• The velocity field $v$ of the fluid is subjected to conservation of mass and momentum (Navier-Stokes equations), but on the field-scale it suffices to consider Darcy’s law, which relates the water flux to the gradient of the pressure $p$. In the momentum balance an electric body force enters, which couples the fluid flow to the electric field and the number densities $c^\pm$.

• The evolution of the number densities $c^\pm$ of the charged particles are governed by transport equations (Nernst-Planck equations), which are nothing but the mass balance equations for the respective particle species. In the mass flux, a convective and electric drift term are respectively present. This couples the number densities $c^\pm$ to the fluid flow and the electric field.

Throughout the rest of the paper, we denote by $I := (0, T)$ an open time interval with end time $T$. Further, we denote by $\Omega$ a bounded domain with boundary $\partial \Omega$ and by $\Omega_T := I \times \Omega$ the time space cylinder with lateral boundary $\partial \Omega_T := I \times \partial \Omega$.

Mathematically, DNPP-systems consist of the following three coupled systems of partial differential equations (1a)-(1h). The preceding explanations point out that all the physically important feedback mechanisms are included in DNPP-systems.

**Poisson equation (Gauss’s law for the electrostatic potential):**

\[-\nabla \cdot (\varepsilon_0 \varepsilon_r D \nabla \Phi) - \sigma = \theta z e (c^+ - c^-) \quad \text{in } \Omega_T, \quad (1a)\]

\[\Phi = 0 \quad \text{on } \partial \Omega_T. \quad (1b)\]

**Extended Darcy’s law:**

\[K^{-1} v = -\mu^{-1} \left( \nabla p + ze(c^+ - c^-) \nabla \Phi \right) \quad \text{in } \Omega_T, \quad (1c)\]

\[\nabla \cdot v = 0 \quad \text{in } \Omega_T \quad (1d)\]

\[v \cdot \nu = 0 \quad \text{on } \partial \Omega_T. \quad (1e)\]

**Nernst-Planck equations:**

\[\theta \partial_t c^\pm + \nabla \cdot \left( v c^\pm - D \nabla c^\pm \mp \frac{ze}{kT} D c^\pm \nabla \Phi \right) = \mp \theta(c^+ - c^-) \quad \text{in } \Omega_T, \quad (1f)\]

\[c^\pm(0) = c^{\pm,0} \quad \text{on } \Omega. \quad (1h)\]

In the above equations, $\varepsilon_0 \varepsilon_r$ is the electric permittivity of the medium, $D$ the diffusivity, $\sigma$ the background charge density, $\theta$ the porosity, $z$ the valency, $e$ the elementary charge, $K$ the permeability of the medium, $\mu$ the dynamic viscosity of the fluid, $k$ the Boltzmann constant, $\tau$ the temperature, and $c^{\pm,0}$ the initial values of the number densities $c^\pm$.

**Remark 2.1** The initial conditions for $\Phi$, $v$, $p$ are not considered as external data. In fact, the initial values $\Phi(0)$, $v(0)$, $p(0)$ are obtained by substituting the initial data $c^{\pm,0}$ in Poisson equation and Darcy’s law.

**Remark 2.2** We focus on the electric interaction and therefore impose linear reaction rates. Other reaction rates that satisfy appropriate structural assumptions, e.g., globally Lipschitz continuous rates, can be handled in a similarly way.
Remark 2.3 Many of the electrokinetic phenomena are based on the interaction between a charged surface and the charged particles. Since a charged surface is modeled either by nonhomogeneous Dirichlet boundary conditions (surface potential) or by nonhomogeneous Neumann boundary conditions (surface charge density), it is not admissible to assume homogeneous boundary conditions for the electrostatic potential $\Phi$. For this reason, it is important to note, that nonhomogeneous Dirichlet boundary conditions are included in our model. This follows by starting with the Poisson equation

$$-\nabla \cdot \left( \epsilon_0 \epsilon_r \nabla \tilde{\Phi} \right) - \tilde{\sigma} = \theta z_e (c^+ - c^-) \quad \text{in } \Omega_T,$$

$$\tilde{\Phi} = g \quad \text{on } \partial \Omega_T.$$

Firstly assume, that $G$ is a function on $\Omega_T$ with $G|_{\partial \Omega_T} = g$ and that, on the other hand, $G$ satisfies certain regularity properties specified later on in Remark 3.1. Secondly, define the shifted potential $\bar{\Phi}$ by $\bar{\Phi} := \tilde{\Phi} - G$ and the shifted background charge density $\sigma$ by $\sigma := \bar{\sigma} - \Delta G$. This leads to the above Poisson equation (1a), (1b) for the shifted potential $\bar{\Phi}$. Furthermore, for nonhomogeneous Neumann boundary conditions, analogue results as presented in this paper are obtained with more technical effort but with the same techniques as well.

Remark 2.4 It is worth noting, that this set of equations is closely related to the widely used Poisson-Boltzmann model in the special case of a fluid at rest, stationary mass transport and vanishing reaction rates, cf. [25].

3 Analysis of the Model

In this section, we show the existence of a global and unique weak solution of the DNPP-system. The main difficulties in the following proofs arise from the nonlinear coupling by means of the product terms in Darcy’s Law and in Nernst-Planck equations. This reflects the fact, that these product terms describe exactly the complicated electrokinetic feedback mechanisms. For the remainder of the paper, the following structural assumptions are supposed to hold.

(G0) Parameter: We set $\epsilon_0 \epsilon_r = \theta = \mu = z_e = k \tau = 1$ since our analytical investigations are not influenced by their respective physical values, but the readability of the following estimates gains a lot. However, the number of parameters in the DNPP-system could also be reduced to a minimum by nondimensionalization.

(G1) Geometry and space dimension: $\Omega \subset \mathbb{R}^3$ is a bounded and convex Lipschitz domain.

(G2) (Initial) Data: For the background charge density $\sigma$ holds $\sigma \in L^{\infty}(\Omega_T)$ and the initial data $c^{\pm,0}$ are nonnegative and bounded, i.e.

$$c^{\pm,0} \in L^{\infty}(\Omega) \quad \text{and} \quad c^{\pm,0}(x) \geq 0 \quad \text{for a.e. } x \in \Omega.$$
(G3) Coefficients: The coefficients are bounded and symmetric positive definite matrices

\[
\sum_{ij} \zeta_i D_{ij} \zeta_j \geq \alpha_D |\zeta|^2 \quad \text{and} \quad \sum_{ij} \eta_i D_{ij} \zeta_j \leq C_D |\eta| |\zeta| \quad \forall \eta, \zeta \in \mathbb{R}^n ,
\]

\[
\sum_{ij} \zeta_i K_{ij}^{-1} \zeta_j \geq \alpha_K |\zeta|^2 \quad \text{and} \quad \sum_{ij} \eta_i K_{ij}^{-1} \zeta_j \leq C_K |\eta| |\zeta| \quad \forall \eta, \zeta \in \mathbb{R}^n .
\]

**Remark 3.1** The assumption \( \sigma \in L^\infty(\Omega_T) \) shows, that in Remark 2.3 the minimal regularity assumption on the function \( G \) is \( G \in L^\infty(0, T; W^{2,\infty}(|\Omega|)) \). Hence, our model contains any nonhomogeneous Dirichlet boundary data, that satisfy this minimal regularity property.

**Remark 3.2** In the case of homogenization results, the definition of the tensors \( D, K^{-1} \) via so-called cell problems can be found in [16]. The symmetry and positive definiteness of such tensors is shown, e.g., in [26].

### 3.1 Weak solutions

First of all, let \( 1 \leq p \leq \infty \) and denote for \( \mathbb{R}^d \)-valued functions the standard Lebesgue spaces by \( L^p(\Omega; \mathbb{R}^d) \), \( L^p(\Omega) := L^p(\Omega; \mathbb{R}) \) and the standard Sobolev spaces by \( W^{1,p}(\Omega; \mathbb{R}^d) \), \( W^{1,p}(\Omega) := W^{1,p}(\Omega; \mathbb{R}) \). Moreover, we set \( H^1(\Omega) := W^{1,2}(\Omega) \). Here, the subscript 0 denotes that we consider only those functions with vanishing traces. The extensions of these function spaces to the time space cylinders \( \Omega_T \), are defined just as in [27], [28]. We additionally define

\[
H_{\text{div},0}^1(\Omega; \mathbb{R}^3) := \{ \varphi \in L^2(\Omega; \mathbb{R}^3), \nabla \cdot \varphi \in L^2(\Omega), \varphi \cdot \nu = 0 \text{ on } \partial \Omega \} .
\]

Multiplying the system of equations (1a), (1c), (1d), (1f) with test functions \( \varphi_2 \in L^2(\Omega), \varphi_3 \in H_{\text{div},0}^1(\Omega; \mathbb{R}^3), \varphi_4 \in L^2(\Omega), \varphi_1 \in H^1(\Omega) \) and integrating by parts, we obtain the following weak formulation of the DNPP-system (1a)–(1g):

\[
\int_\Omega \nabla \cdot (D \nabla \Phi) \varphi_2 \, dx = \int_\Omega (c^+ - c^- + \sigma) \varphi_2 \, dx ,
\]

\[
\int_\Omega K^{-1} v \cdot \varphi_3 \, dx = \int_\Omega p \nabla \cdot \varphi_3 - (c^+ - c^-) \nabla \Phi \cdot \varphi_3 \, dx ,
\]

\[
\int_\Omega (\nabla \cdot v) \varphi_4 \, dx = 0 ,
\]

\[
(\partial_t c^\pm, \varphi_1)_{H_{\text{div},0}^1(\Omega),H^1(\Omega)} + \int_\Omega \left( -vc^\pm + D \nabla c^\pm \pm D \nabla \Phi \right) \cdot \nabla \varphi_1 \, dx = \int_\Omega (c^+ - c^-) \varphi_1 \, dx .
\]
Definition 3.3 The vector \((\Phi, v, p, e^+, e^-) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}\) is a weak solution of the DNPP-systems (1a)–(1h), if
\[
\begin{align*}
\Phi(t, x) &\in L^\infty(0, T; H^2_0(\Omega)) \quad \text{solves (2a) for all } \varphi_2 \in L^2(\Omega), \\
v(t, x) &\in L^2(0, T; H^1(\Omega; \mathbb{R}^3)) \quad \text{solves (2b) for all } \varphi_3 \in H^1(\Omega; \mathbb{R}^3), \\
p(t, x) &\in L^2(0, T; L^2(\Omega)/\mathbb{R}) \quad \text{solves (2c) for all } \varphi_4 \in L^2(\Omega), \\
c^\pm(t, x) &\in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \\
&\text{with } \partial_t e^\pm \in L^2(0, T; H^{-1}(\Omega)) \quad \text{solves (2d) for all } \varphi_1 \in H^1_0(\Omega),
\end{align*}
\] and if \(e^\pm\) take their initial values in the sense that
\[
\langle e^+(0) - e^+, \varphi \rangle_{L^2(\Omega)} = 0 \quad \text{for all } \varphi \in L^2(\Omega).
\]

3.2 Nonnegativity and Boundedness

In this section, we prove that the number densities \(e^\pm\) are physically reasonable quantities.

Theorem 3.4 Let \((\Phi, v, p, e^+, e^-)\) be a weak solution of (1a)–(1h) according to Definition 3.3. Then, the number densities are nonnegative, i.e.
\[
e^\pm(t, x) \geq 0 \quad \text{for a.e. } t \in (0, T), \ x \in \Omega.
\]

Proof. We test the Nernst-Planck equations (2d) with \(\varphi_1 = e^\pm := \min(0, e^\pm).\) This yields after summation over \(\pm\)
\[
\begin{align*}
\sum_{\pm} \frac{d}{dt} ||e^\pm||_{L^2(\Omega)}^2 + \sum_{\pm} \int_\Omega \nabla c^\pm \cdot \nabla c^\pm \, dx - \sum_{\pm} \frac{1}{2} \int_\Omega v \cdot (c^\pm)^2 \, dx &= - \sum_{\pm} \frac{1}{2} \int_\Omega \nabla \Phi \cdot \nabla (c^\pm)^2 \, dx + \sum_{\pm} \int_\Omega \theta (c^+ - c^-) c^\pm \, dx \\
&\iff \quad I + II - III = -IV + V.
\end{align*}
\]

We estimate integral \(II\) by applying \((G2)\) and rewrite the integrals \(III\) and \(IV\) by using integration by parts. Afterwards, we substitute (2a) in \(IV\) and get \(III = 0\) by taking (1d) and (1e) for the velocity field \(v\) into account. By standard calculations, we arrive with \(V \leq 0\) at
\[
\begin{align*}
\sum_{\pm} \frac{d}{dt} ||e^\pm||_{L^2(\Omega)}^2 + \alpha_D ||\nabla c^\pm||_{L^2(\Omega)}^2 &
\leq - \frac{1}{2} \int_\Omega (c^+ - c^-) ((c^+)^2 - (c^-)^2) \, dx + \frac{1}{2} ||\sigma||_{L^\infty(\Omega)} \sum_{\pm} ||c^\pm||_{L^2(\Omega)}^2 \\
&\leq \frac{1}{2} ||\sigma||_{L^\infty(\Omega)} \sum_{\pm} ||c^\pm||_{L^2(\Omega)}^2.
\end{align*}
\]

In the last step, we used a sign condition. Together with Gronwall’s Lemma, cf. [27, Appendix B], this concludes the proof. \(\square\)
Theorem 3.5 Let \((\Phi, v, p, c^+, c^-)\) be a weak solution of (1a)–(1h) according to Definition 3.3. Then the number densities \(c^\pm\) are bounded in \(L^\infty(\Omega_T)\). Furthermore, their \(L^\infty(\Omega_T)\)-norms are estimated by their \(L^2(\Omega_T)\)-norm via

\[
\sum_{\pm} \|c^\pm\|_{L^\infty(\Omega_T)} \leq C_M \sum_{\pm} \|c^\pm\|_{L^2(\Omega_T)} + 4 \sum_{\pm} \|c^{\pm,0}\|_{L^\infty(\Omega)} + 4.
\]

The crucial step in Moser’s iteration technique consists in testing the Nernst-Planck equation (2a) with the test function \(\varphi = (c^+ - K)^{2\alpha} (c^- - K)_+ \in H_0^1(\Omega)\) for arbitrary \(\alpha \geq 0\) and estimating the integrals \(I^\pm - VI^\pm\):

The convection terms \(II^\pm\) in (4) vanish after integration by parts and applying (1d), (1e), i.e.

\[
II^\pm = \int_{\Omega} (c^\pm - K) \cdot \nabla \left( (c^+_M - K)^{2\alpha} (c^- - K)_+ \right) \, dx = 0.
\]

For the diffusion terms \(III^\pm\) in (4), we obtain with (G2)

\[
III^\pm \geq \frac{2\alpha D\alpha}{(\alpha + 1)^2} \|\nabla (c^+_M - K)^{\alpha + 1}\|_{L^2(\Omega)}^2 + \alpha D \| (c^+_M - K)^{\alpha} \nabla (c^- - K)_+ \|_{L^2(\Omega)}^2.
\]

We use again integration by parts for the electric drift terms \(IV^\pm\) in (4) and substitute Poisson equation (2a). This leads to

\[
IV^\pm = \pm \int_{\Omega} (c^+ - c^-) \left( (c^+_M - K)^{2\alpha} (c^- - K)_+ + \frac{\alpha}{\alpha + 1} (c^+_M - K)^{2\alpha + 2} \right) \, dx
\]

\[
= \pm (IV_{1,1}^\pm + IV_{1,2}^\pm).
\]
Taking the sum over ±, we note that with $A_1$ and $A_2$ defined by

\[
A_1 := \left( c^+ - K - (c^- - K) \right), \\
A_2 := \left[ \left( (c_M^+ - K)^{2\alpha} - (c_M^- - K)^{2\alpha} \right) \\
+ \frac{\alpha}{\alpha + 1} \left( (c_M^+ - K)^{2\alpha + 2} - (c_M^- - K)^{2\alpha + 2} \right) \right],
\]

it holds $A_1 A_2 \geq 0$. Hence, for the sum of $IV.1^\pm$, we get immediately

\[
\sum_{\pm} IV.1^\pm = IV.1^+ - IV.1^- = \int_{\Omega} A_1 A_2 \, dx \geq 0.
\]

The remaining integrals $IV.2^\pm$ are estimated by

\[
\sum_{\pm} |IV.2^\pm| \leq \|\sigma\|_{L^\infty(\Omega_T)} \left[ 1 + \frac{\alpha}{\alpha + 1} \right] \sum_{\pm} \left( \| (c_M^+ - K)^\alpha \|_{2} \right)
\]

For the terms of lower order $VI^\pm$ in (4), we introduce a new level $h$ with $K > h > K_0$ and arrive at

\[
VI^\pm \leq \|\sigma\|_{L^\infty(\Omega_T)} \frac{K}{K - h} \| (c_M^+ - h)^\alpha (c_M^+ - h) \|_{2}.
\]

In order to estimate the sum of the reaction terms $V^\pm$ in (4), we take into account that according to (5) it holds $\bar{K} > 1$, which ensures $1 - K \leq 0$. This implies

\[
\sum_{\pm} V^\pm \leq |1 - K| \sum_{\pm} \int_{\Omega} \left( (c_M^+ - K)^\alpha (c_M^+ - K) \right) dx
\]

\[
- |1 - K| \sum_{\pm} \int_{\Omega} \left( (c_M^+ - K)^{2\alpha} (c_M^+ - K) \right) dx
\]

\[
=: V.1 + V.2.
\]

Note that in V.2 the last factor has superscript $\mp$. Analyzing the sign and superscripts $\pm$ and $\mp$ and using again the level $h$ with $K > h > K_0$, we deduce, in summary, for the reaction terms

\[
\sum_{\pm} V^\pm \leq |1 - K| \left( 1 + \frac{K}{K - h} \right) \sum_{\pm} \| (c_M^+ - h)^\alpha (c_M^+ - h) \|_{2}.
\]

Finally, we combine the above estimates for the integrals $II^\pm - VI^\pm$ and integrate (4) with respect to time. Furthermore, we replace on the right hand side the level $K$ by the level $h$, which is admissible as it holds $K_0 < h < K$. This leads us to the desired “truncated” energy
estimate
\[
\sum_{\pm} \int_0^T \left( \partial_t (c^\pm - K), \frac{2\alpha}{(\alpha + 1)^2} \| \nabla (c_M^\pm - K) \|_{L^2(\Omega_T)}^{\alpha + 1}, \right)_{H^{-1}(\Omega), H_0^1(\Omega)} dt \\
+ \sum_{\pm} \alpha_D \| (c_M^\pm - K)_+ \|_{L^2(\Omega_T)}^{\alpha} \| \nabla (c^\pm - K)_+ \|_{L^2(\Omega_T)}^2 \\
\leq (|1 - K| + \| \sigma \|_{L^\infty(\Omega_T)} (1 + \frac{K}{K - h}) \sum_{\pm} \| (c_M^\pm - h)_+ \|_{L^2(\Omega_T)}^{\alpha} \| (c^\pm - h)_+ \|_{L^2(\Omega_T)}^{\alpha + 1} \\
+ \| \sigma \|_{L^\infty(\Omega_T)} \alpha \| (c_M^\pm - h)_+ \|_{L^2(\Omega_T)}^{\alpha + 1} \| (c^\pm - h)_+ \|_{L^2(\Omega_T)}^{\alpha + 1}.
\]
\( (6) \)

In order to start Moser’s iteration procedure, we define for \( j \in \mathbb{N}_0 \) a sequence of exponents \( \alpha_j \) and levels \( K_j \) by
\[
1 + \alpha_j := \left( \frac{n + 2}{n} \right)^j = (5/3)^j, \quad \text{since we consider } n = 3, \text{ cf. } (G1), \quad (7)
\]
\[
h := K_j := K_0 + (1 - 2^{-j})K_0 \quad \text{with } K_0 \text{ from } (5), \quad K := K_{j+1}.
\]

Additionally, we repeat that the \( L^p \left( (0, T) \times \Omega; \mathbb{R}^2 \right) \)-norms, \( 1 \leq p \leq \infty \), of the number density vector \((c^+, c^-) \in \mathbb{R}^2\) are defined by
\[
\|(c^+, c^-)\|_{L^p(\Omega_T, \mathbb{R}^2)} := \left( \sum_{\pm} \| c^\pm \|_{L^p(\Omega_T)}^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty . \quad (8)
\]
\[
\|(c^+, c^-)\|_{L^\infty(\Omega_T, \mathbb{R}^2)} := \sum_{\pm} \| c^\pm \|_{L^\infty(\Omega_T)} , \quad p = \infty .
\]

Moser’s iteration over \( j \in \mathbb{N}_0 \) consists basically in two steps: Firstly, we establish the energy estimate \( (6) \) for the exponent \( \alpha_j \) and secondly, we transform this energy estimate in combination with a Sobolev’s embedding theorem to higher integrability of the number densities to the exponent \( \alpha_{j+1} \). Then, we start again with the energy estimate \( (6) \) for the higher exponent \( \alpha_{j+1} \).

**Step 1 (base step):** We start the induction with \( j = 0 \) and note that it holds \( 1 + \alpha_0 = 1 \).
Substituting these choices in \( (6) \), we notice that due to \( (5) \) and \( (7) \) the initial values of \( c^\pm - \)
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\[ \sum_{\pm} \max_{0 \leq t \leq T} \left\| (c^\pm - K_1)_{\pm} \right\|_{L^2(\Omega)} + \sum_{\pm} \left\| \nabla(c^\pm - K_1)_{\pm} \right\|_{L^2(\Omega_T)} \]

\[ \leq \frac{2}{\alpha_D} \left( |1 - K_1| + \|\sigma\|_{L^\infty(\Omega_T)} \right) \left( 1 + \frac{K_1}{K_1 - K_0} \right) \sum_{\pm} \left\| (c^\pm - K_0)_{\pm} \right\|_{L^2(\Omega_T)}^2 \]

\[ =: C_0 \sum_{\pm} \left\| (c^\pm - K_0)_{\pm} \right\|_{L^2(\Omega_T)}^2. \]  

(9)

Next, we transform the above inequality with the parabolic Sobolev’s embedding theorem\(^1\), cf. [30, Proposition 3.1] and \(1 + \alpha_j + 1 = 1 + \alpha_1 = 5/3\) and (8) to

\[ \left\| ((c^+ - K_1)_{+}, (c^- - K_1)_{+}) \right\|_{L^2(1+\alpha_1)(\Omega_T;\mathbb{R}^2)} = \left[ \sum_{\pm} \left\| (c^\pm - K_1)_{\pm} \right\|_{L^2(\Omega_T)}^{2(5/3)} \right]^{\frac{2}{5}} \]

\[ \leq (C_S)^{\frac{1}{2}} \left[ \sum_{\pm} \max_{0 \leq t \leq T} \left\| (c^\pm - K_1)_{\pm} \right\|_{L^2(\Omega)}^2 + \left\| \nabla(c^\pm - K_1)_{\pm} \right\|_{L^2(\Omega_T)}^2 \right]^{\frac{1}{2}} \]

\[ \leq (C_SC_0)^{\frac{1}{2}} \left[ \sum_{\pm} \left\| (c^\pm - K_0)_{\pm} \right\|_{L^2(\Omega_T)}^2 \right]^{\frac{1}{2}}. \]

**Step 2 (induction hypotheses):** For an \( J \in \mathbb{N} \), we assume that the induction hypothesis holds with the exponent \( 1 + \alpha_{j-1} \), from (7), i.e.

\[ \left\| ((c^+ - K_J)_{+}, (c^- - K_J)_{+}) \right\|_{L^2(1+\alpha_J)(\Omega_T;\mathbb{R}^2)} \]

\[ \leq \prod_{j=0}^{J-1} (C_SC_J)^{\frac{1}{2}} \left[ \sum_{\pm} \left\| (c^\pm - K_0)_{\pm} \right\|_{L^2(\Omega_T)}^2 \right]^{1/2}. \]

Hereby, the constant \( C_S \) results from Sobolev’s embedding theorem and \( C_j \) is defined according to (6) by

\[ C_j := (1 + \alpha_j)^2 \frac{2}{\alpha_D} \left( |1 - K_J| + \|\sigma\|_{L^\infty(\Omega_T)} \right) \left( 1 + \frac{K_j}{K_j - K_{j-1}} \right) \]

\[ + \alpha_j (1 + \alpha_j) \frac{2}{\alpha_D} \|\sigma\|_{L^\infty(\Omega_T)} . \]  

(10)

**Step 3 (induction step):** We choose \( 1 + \alpha_J = (3/5)^J \) in (6) and consider the limit of \( M \to \infty \). We note that due to (5) and (7) the initial values of \((c^\pm - K_{j+1})_{+}\) vanish. Thereby,

\[ \text{The constant from Sobolev’s embedding theorem is in the following denoted by } C_S. \]
Step 4: Together with \( \| (c^+, c^-) \|_{L^\infty(\Omega_T; \mathbb{R}^2)} = \lim_{{p \to \infty}} \| (c^+, c^-) \|_{L^p(\Omega_T; \mathbb{R}^2)} \), we get from (11) with \( J \to \infty \) immediately

\[
\| (c^+ - K_\infty, c^- - K_\infty) \|_{L^\infty(\Omega_T; \mathbb{R}^2)} \\
\leq \prod_{j=0}^{J} \left( C_S C_J \right)^{\frac{1}{2}} \left[ \sum_{\pm} \| (c^\pm - K_0)^+ \|_{L^2(\Omega_T)} \right]^{1/2}.
\]

This is exactly the desired result, which allows us to control arbitrary higher norms of the number densities \( c^\pm \) in terms of their \( L^2 \)-norm. Hence, we deduced that the number densities \( c^\pm \) are higher integrable for arbitrary large \( p \in [1, \infty) \).

In combination with the induction hypothesis, we obtain according to the above estimate directly

\[
\| (c^+ - K_{J+1}), (c^- - K_{J+1}) \|_{L^{2(1+\alpha_J)}(\Omega_T; \mathbb{R}^2)} \\
\leq \prod_{j=0}^{J} \left( C_S C_J \right)^{\frac{1}{2}} \left[ \sum_{\pm} \| (c^\pm - K_0)^+ \|_{L^2(\Omega_T)} \right]^{1/2}. \tag{11}
\]

The right hand side remains bounded as presupposed in Step 2 and hence taking the limit \( M \to \infty \) was admissible. We now apply again Sobolev’s embedding theorem, cf. [30, Proposition 3.1], use the notation in (8), and deduce from the above inequality

\[
\| (c^+ - K_{J+1}), (c^- - K_{J+1}) \|_{L^{2(1+\alpha_J)}(\Omega_T; \mathbb{R}^2)} \\
= \left[ \sum_{\pm} \int_0^T \int_{\Omega} \| (c^\pm - K_{J+1})_{1+\alpha_J} \|_{L^2(\Omega_T)} \right]^{\frac{1}{2}} \frac{1}{\Gamma(\frac{1}{1+\alpha_J})} \\
\leq \left[ C_S \sum_{\pm} \max_{0 \leq t \leq T} \| (c^\pm - K_{J+1})_{1+\alpha_J} \|_{L^2(\Omega_T)} \right]^{\frac{1}{2}} \frac{1}{\Gamma(\frac{1}{1+\alpha_J})} \\
\leq \left( C_S C_J \right)^{\frac{1}{2}} \left[ \sum_{\pm} \| (c^\pm - K_J)^+ \|_{L^{2(1+\alpha_J)}(\Omega_T)} \right]^{\frac{1}{2}} \\
= \left( C_S C_J \right)^{\frac{1}{2}} \left[ \sum_{\pm} \| (c^\pm - K_J)^+ \|_{L^{2(1+\alpha_J)}(\Omega_T)} \right]^{\frac{1}{2}}.
\]

Thus, we have established the \( L^\infty \)-bound of \( c^\pm \) in case the infinite product converges. This statement is easily shown by writing the infinite product as infinite series and using the
root test. We conclude the proof by inserting the definitions of \(K_0\), cf. (5), and of \(K_\infty = \lim_{J \to \infty} K_J\), cf. (7).

**Remark 3.6** By means of (9), we obtain in case of \(K_0 = K_1 = 0\) directly the standard parabolic estimate in its common form

\[
\frac{d}{dt} \sum_{\pm} ||c^\pm||_{L^2(\Omega)}^2 + \sum_{\pm} ||\nabla c^\pm||_{L^2(\Omega)}^2 \leq \frac{2}{\alpha_D} \left\| \sigma \right\|_{L^\infty(\Omega_T)} \sum_{\pm} ||c^\pm||_{L^2(\Omega)}^2.
\]

(12)

**Remark 3.7** The preceding proof shows that the constant \(C_M\) in the \(L^\infty\)-estimate (3) depends on \(C_S\) and \(C_j\). The constant \(C_S\), resulting from the parabolic Sobolev embedding theorem, is given in [30, Proposition 3.1] in the case of homogeneous Dirichlet boundary conditions by \(C_S = C_S(\partial \Omega)\). On the other hand, the constants \(C_j\) are defined explicitly in (10). Hence, the constant \(C_M\) in the above \(L^\infty\)-estimate is independent of the end time \(T\) and consequently the \(L^\infty\)-estimate continues to hold even for \(T = \infty\).

**Remark 3.8** The proof of Theorem 3.5 is valid for arbitrary space dimension, i.e. \(\Omega \subset \mathbb{R}^n\), \(n \geq 2\). For this reason, the stated boundedness in Theorem 3.5 holds even for \(n \geq 2\). We restrict ourselves to \(n = 3\), since in the following fixed point approach, we use compact embeddings according to Aubin-Lions lemma, cf. [28, Lemma 7.7], which are valid only for \(n \leq 3\).

### 3.3 Existence

In this section, we proof the existence of global weak solutions of the DNPP-system. Our proof is based on the following version of Schauder’s fixed point theorem, see [31, Corollary 9.6].

**Theorem 3.9** Let \(F : K \subset X \to K\) be continuous, where \(K\) is a nonempty, compact, and convex set in a locally convex space \(X\). Then, \(F\) has a fixed point.

First, we show local existence with the aid of the above fixed point theorem. The global existence then follows by extension of the local result with bootstrapping arguments.

**Theorem 3.10** There exists a global weak solution \((\Phi, v, p, c^+, c^-)\) of (1a)–(1h) according to Definition 3.3.

**Proof.** For the application of the above fixed point theorem, Theorem 3.9, we first of all define the appropriate space \(X\) by

\[
X := L^\infty(I; L^2(\Omega; \mathbb{R}^2)) \cap L^2(I; \mathcal{H}^1_0(\Omega; \mathbb{R}^2)) \cap H^1(I; H^{-1}(\Omega; \mathbb{R}^2)) \cap L^\infty(\Omega_T; \mathbb{R}^2),
\]

\[
\| \cdot \|_X := \| \cdot \|_{L^\infty(I; L^2(\Omega; \mathbb{R}^2))} + \| \cdot \|_{L^2(I; \mathcal{H}^1_0(\Omega; \mathbb{R}^2))} + \| \cdot \|_{H^1(I; H^{-1}(\Omega; \mathbb{R}^2))} + \| \cdot \|_{L^\infty(\Omega_T; \mathbb{R}^2)}.
\]

According to this definition, \(X\) is interpreted twofold in the sense that

- \((X, \text{ weak}^*)\) is a locally convex space. Here, \(X\) is supplemented with the weak*-topology.
- \((X, \| \cdot \|_X)\) is a Banach space. Here, \(X\) is supplemented with the norm-topology.
In the following, we consider $X$ as locally convex space $(X, \text{weak}^*)$ and all topological terms refer to the weak*--topology. Furthermore, the predual $X_0$ of $X$ is given by (see [32, Chapter I, IV])

$$X_0 := L^1(I; L^2(\Omega; \mathbb{R}^2)) + L^2(I; H^{-1}(\Omega; \mathbb{R}^2)) + H^{-1}(I; H^1(\Omega; \mathbb{R}^2)) + L^1(\Omega_T; \mathbb{R}^2).$$

Hence, $X_0$ is a separable Banach space with dual $X$ and the topological terms for $(X, \text{weak}^*)$ based on sets are equivalent with those based on sequences, cf. [31, 32]. In particular, the notion of weak*--continuous/compact is equal to sequentially weak*--continuous/compact.

**Step 1 (Definition of the fixed point set $K$):** For $R > 0$, we define $K$ by

$$K := \{ \varphi = (\varphi_1, \varphi_2) \in X : \|\varphi\|_X \leq R \} \subset X.$$ 

$K$ is obviously nonempty for any $R > 0$. $K$ is convex by definition and $K$ is weak*--compact according to Banach-Alaoglu-Bourbaki theorem, cf. [28, Theorem 1.7].

**Step 2 (Definition of the fixed point operator $F$):** Define $F$ as

$$F := F_3 \circ F_2 \circ F_1 : K \subset X \rightarrow X.$$ 

The suboperators $F_1, F_2, F_3$ are defined as follows:

(i) $F_1 : X \rightarrow X \times L^\infty(I; H^2_0(\Omega)) =: Y,$

$$\left(\tilde{c}^+, \tilde{c}^-\right) \mapsto \left(\tilde{c}^+, \bar{c}^-, \Phi\right),$$

with $\Phi$ being solution of $(2a)$ for given data $\tilde{c}^+$, i.e., for all $\varphi_2 \in L^2(\Omega)$, it holds

$$\int_{\Omega} \nabla \cdot (D\nabla \Phi) \varphi_2 \, dx = \int_{\Omega} \left(\bar{c}^-_0 - \tilde{c}^+_0 + \sigma\right) \varphi_2 \, dx. \quad (13)$$

$F_1$ is well defined, since there exists a unique solution $\Phi \in L^\infty(I; H^2(\Omega))$ of $(13)$. For this well-known result see, e.g., [33, Theorem 5.1.2] and notice that $t$ plays only the role of a parameter.

(ii) $F_2 : Y \rightarrow Y \times L^\infty(H^1_{\text{div},0}(\Omega; \mathbb{R}^3)) \times L^\infty(I; L^2(\Omega)/\mathbb{R}) =: Z,$

$$(v, p, \Phi) \mapsto (\bar{c}^+, \bar{c}^-, \Phi, v, p),$$

with $(v, p)$ being solutions of $(2b), (2c)$ for given data $\bar{c}^\pm$ and $\Phi$, i.e., for all $\varphi_3 \in H^1_{\text{div},0}(\Omega; \mathbb{R}^3)$ and $\varphi_4 \in L^2(\Omega)$, it holds

$$\int_{\Omega} K^{-1}v \cdot \varphi_3 \, dx + \int_{\Omega} p \nabla \cdot \varphi_3 \, dx = -\int_{\Omega} (\bar{c}^+ - \bar{c}^-) \nabla \Phi \cdot \varphi_3 \, dx, \quad (14)$$

$$\int_{\Omega} (\nabla \cdot v) \varphi_4 \, dx = 0. \quad (15)$$

Using the regularities of $\bar{c}^\pm$ and $\nabla \Phi$ together with [33, Theorem 7.4.1] provides directly the existence of a unique weak solution $(v, p)$ of $(14), (15)$ in the stated spaces. However, the pressure field $p$ is only determined up to a constant. Imposing zero mean value, i.e. $\int_{\Omega} p \, dx = 0$, leads to uniqueness of $p$ in $L^2(\Omega)/\mathbb{R}$ and hence the operator $F_2$ is well defined.
(iii) \( F_3 : Z \rightarrow X \),

\[
(\bar{c}^+, \bar{c}^-, \Phi, v, p) \mapsto (c^+, c^-),
\]

with \( c^\pm \) being solution of (2d) for given data \( \Phi, v, p \), i.e., for all \( \varphi_2 \in H^1_0(\Omega) \), it holds

\[
\langle \partial_t c^\pm, \varphi_2 \rangle_{H^{-1}(\Omega) \times H^1_0(\Omega)} + \int_{\Omega} (-v c^\pm + D\nabla c^\pm \pm D\Phi) \cdot \nabla \varphi_2 \, dx
\]

\[
\Rightarrow I^\pm + II^\pm + III^\pm + IV^\pm = V^\pm. \tag{16}
\]

Applying Rothe’s method, cf. [28, Chapter 8.2], taking the regularities of \( \Phi, v, p \) and Theorem 3.4, Theorem 3.5 into account guarantees the existence of an unique weak solution \((c^+, c^-) \in X\) of (16) and shows that \( F_3 \) is well defined.

We note, that the fixed point operator \( F \) is solely a function of the concentrations \( c^\pm \). For this reason, a fixed point \((c^+, c^-)\) of \( F \) only solves the respective transport equations for \( c^\pm \). However, the suboperators \( F_1 \) and \( F_2 \) contain the necessary information about the electrostatic potential \( \Phi \), the velocity field \( v \), and the pressure \( p \). Furthermore, by means of stability estimates similar to the inequalities in the proof of Theorem 3.11, it becomes clear, that the existence of a fixed point \((c^+, c^-)\) shows the existence of solutions \( \Phi \) and \((v, p)\) of the nonlinear system (1a)-(1e) as well. Hence, it is admissible to focus on the concentrations \( c^\pm \).

Step 3 (self mapping property \( F(K) \subset K \)): Let \((\bar{c}^+, \bar{c}^-) \in K\), i.e., \( \|c^\pm\|_X \leq R \) hold. Then, we obtain the following a priori estimates:

(i) For Poisson equation (13), we use \( \Phi \) as test function and deduce

\[
\|\nabla \Phi\|_{L^2(\Omega_T)} \leq \frac{2}{\alpha_D} \|c^+ - c^- + \sigma\|_{L^2(\Omega_T)} \leq \frac{2|\Omega|}{\alpha_D} (2R + \|\sigma\|_{L^\infty(\Omega_T)}) \left( \frac{T}{4} \right)^{\frac{1}{2}} =: C_1. \tag{17}
\]

(ii) Testing Darcy’s law (14) with \( v \) gives

\[
\|v\|_{L^2(I;H^1_{\text{div,0}}(\Omega;\mathbb{R}^3))} \leq \frac{1}{\alpha_K} \|\bar{c}^+_0 - \bar{c}^-_0\|_{L^\infty(\Omega_T)} \|\nabla \Phi\|_{L^2(\Omega_T)} = \frac{2RC_1}{\alpha_K} \left( \frac{T}{4} \right)^{\frac{1}{2}}. \tag{18}
\]

Proceeding as in [33, Chapter 7.2.2], allows to estimate the \( L^2 \)-norm of the pressure \( p \). In summary, this yields

\[
\|v\|_{L^2(I;H^1_{\text{div,0}}(\Omega;\mathbb{R}^3))} + \|p\|_{L^2(\Omega_T)} \leq C_3(\alpha_K, C_1, R) \left( \frac{T}{4} \right)^{\frac{1}{2}}. \tag{19}
\]

(iii) We test the Nernst-Planck equations (16) with \( c^\pm \) and recover (12). Gronwall’s lemma implies

\[
\sum_{\pm} \|c^\pm\|^2_{L^\infty(I;L^2(\Omega))} \leq \exp(C_E T) |\Omega| \sum_{\pm} \|c^{\pm,0}\|^2_{L^\infty(\Omega)} := C_4(T).
\]
Furthermore, after integration with respect to time, we get from (12) together with Poincaré’s inequality\(^2\), cf. [27]

\[
\sum_{\pm} \| \varepsilon^{\pm} \|^2_{L^2(I; H^1_0(\Omega))} \leq T(C_F + C_E)C_4 + \sum_{\pm} \| \varepsilon^{\pm,0} \|^2_{L^2(\Omega)} =: C_5(T).
\]

Since (3) holds for a solution of (16), we arrive in combination with the above inequalities at

\[
\sum_{\pm} \| \varepsilon^{\pm} \|_{L^\infty(\Omega_T)} \leq C_M C_5 + 4 \sum_{\pm} \| \varepsilon^{\pm,0} \|_{L^\infty(\Omega)} + 4 := C_6(T).
\]

We estimate the time derivative by means of (16) and the preceding a priori estimates. This implies

\[
\| \partial_t \varepsilon^{\pm} \|^2_{L^2(I; H^{-1}(\Omega))} \leq \| \varepsilon^{\pm} \|^2_{L^\infty(\Omega_T)} \| v \|^2_{L^2(\Omega_T)} + C_D \| \nabla \varepsilon^{\pm} \|^2_{L^2(\Omega_T)}
+ C_D \| v \|^2_{L^\infty(\Omega_T)} \| \nabla \Phi \|^2_{L^2(\Omega_T)} + \| \varepsilon^{+} - \varepsilon^{-} \|^2_{L^2(\Omega_T)}
\leq C_6 C_2 T^{\frac{1}{2}} + C_D C_5 + C_D C_5 C_1 T^{\frac{1}{2}} + 2 C_5.
\]

As a consequence, every component of the \(X\)-norm of \(\varepsilon^{\pm}\) is controlled in terms of the data, the radius \(R\) and the end time \(T\). Observe that \(C_D, C_4, C_5\) and \(C_6\) are independent of \(R\). Hence, we choose \(R\), e.g., twice as big as the sum consisting of all constants in the a priori estimates, which are independent of \(R\). In the remaining constants, that depend on \(R\), we choose the end time \(T\) sufficiently small such that these contributions nearly vanish. Thus it holds

\[
\| (\varepsilon^+, \varepsilon^-) \|_X = \| \mathcal{F}(\varepsilon^+, \varepsilon^-) \|_X < R.
\]

We note that the above arguments lead to a trade-off between the end time and the radius \(R\) and are the reason that we are restricted to local existence so far. This drawback will be overcome in Step 5.

**Step 4 (continuity):** To conclude the proof of the local existence result, the fixed point operator \(\mathcal{F}\) has to be weak*-continuous. Hereby, we use the already mentioned equivalence between sequentially weak*-continuous and weak*-continuous. Assume that

\[
(\varepsilon_k^+, \varepsilon_k^-) \rightharpoonup (\varepsilon^+, \varepsilon^-) \quad \text{in} \; X.
\]

Since \(\mathcal{F}(\varepsilon_k^+, \varepsilon_k^-) = (\varepsilon_k^+, \varepsilon_k^-)\) is a solution of (16), the self mapping property from Step 3 holds and implies

\[
\| (\varepsilon_k^+, \varepsilon_k^-) \|_X \leq R.
\]

Thus, a subsequence, denoted again by \((\varepsilon_k^+, \varepsilon_k^-)\), weak*-converges to a unique limit \((\varepsilon^+, \varepsilon^-) \in X\). Provided that \((\varepsilon^+, \varepsilon^-)\) solves the “limit” PDE (16), generated by \((\varepsilon^+, \varepsilon^-)\), the operator \(\mathcal{F}\) is indeed weak*-continuous, since it holds

\[
(\varepsilon_k^+, \varepsilon_k^-) \rightharpoonup (\varepsilon^+, \varepsilon^-) \implies (\varepsilon_k^+, \varepsilon_k^-) = \mathcal{F}(\varepsilon_k^+, \varepsilon_k^-) \rightharpoonup \mathcal{F}(\varepsilon^+, \varepsilon^-) = (\varepsilon^+, \varepsilon^-) \quad \text{in} \; X.
\]

\(^2\) Denote the constant from Poincaré’s inequality by \(C_P\).

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To this end, note that due to Aubin-Lions lemma, cf. [28, Lemma 7.7], in particular, the norm-convergences
\[ c_k^\pm \to \bar{c}^\pm \quad \text{in} \quad L^2(\Omega_T) \quad \text{and} \quad c_k^\pm \to \bar{c}^\pm \quad \text{in} \quad L^2(I; L^3(\Omega)) \]  
(20)
hold true. We now consider the respective integrals in (16). The first integral \( I^\pm \) in (16) converges since \( c_k^\pm \) is a priori bounded in \( L^2(I; H^{-1}(\Omega)) \) and hence it holds
\[
\int_0^T \langle \partial_t (c_k^\pm - c^\pm), \varphi \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \, dt = \langle \partial_t (c_k^\pm - c^\pm), \varphi \rangle_{L^2(I; H^{-1}(\Omega)), L^2(I; H^1_0(\Omega))} \to 0.
\]
The diffusion integrals \( II^\pm \) in (16) converge as \( c_k^\pm \) is a priori bounded in \( L^2(0, T; H^1(\Omega)) \), i.e. \( \nabla c_k^\pm \) is bounded in \( L^2(\Omega_T) \). Thus, we arrive at
\[
\int_0^T \int_\Omega D\nabla (c_k^\pm - c^\pm) \cdot \nabla \varphi \, dx \, dt = \langle \nabla (c_k^\pm - c^\pm), \nabla \varphi \rangle_{L^2(\Omega_T), L^2(\Omega_T)} \to 0.
\]
The electric drift integrals \( IV^\pm \) in (16) are estimated by
\[
\int_0^T \int_\Omega (c_k^\pm D\nabla \Phi_k - c^\pm D\nabla \Phi) \cdot \nabla \varphi \, dx \, dt
\leq C_D \| c_k^\pm \|_{L^\infty(\Omega_T)} \| \nabla (\Phi_k - \Phi) \|_{L^2(\Omega_T)} \| \varphi \|_{L^2(\Omega_T)} + \int_0^T \int_\Omega (c_k^\pm - c^\pm) D\nabla \Phi \cdot \nabla \varphi \, dx \, dt.
\]  
(21)
\]
The first term on the right hand side in (21) vanishes since (17) and (20) give
\[
\| \nabla (\Phi_k - \Phi) \|_{L^2(\Omega_T)} \leq C \| (c_k^\pm - \bar{c}^\pm) - (\bar{c}_k^+ - \bar{c}_k^-) \|_{L^2(\Omega_T)} \to 0.
\]
For the second term on the right hand side in (21), we observe that both \( D\nabla \Phi \cdot \nabla \varphi \in L^1(\Omega_T) \) and \( c_k^\pm - c^\pm \in L^\infty(\Omega_T) = (L^1(\Omega_T))^* \) are bounded and therefore are weak*-convergent, i.e., it holds
\[
\int_0^T \int_\Omega (c_k^\pm - c^\pm) D\nabla \Phi \cdot \nabla \varphi \, dx \, dt = \langle (c_k^\pm - c^\pm), D\nabla \Phi \cdot \nabla \varphi \rangle_{L^1(\Omega_T)^*, L^1(\Omega_T)} \to 0.
\]
The convergence of the convection integrals \( II^\pm \) in (16) are shown analogously to the one of the drift term, except for using (18) instead of (17). Furthermore, we note that the subsequences were arbitrary and therefore the whole sequences converge.

**Step 5 (global solution):** Subsequently, we consider an arbitrary large time interval \([0, T]\), which is decomposed by \( 0 =: T_0 < T_1 < \ldots < T_K := T \) into \( K \)-subintervals \([T_i, T_{i+1}]\), \( i \in \{0, \ldots, K-1\} \) such that on each time interval \([T_i, T_{i+1}]\) Step 1 - 4 are fulfilled. Hence, a local solution \( c_i^\pm \) exists on the \( i \)-th time interval \([T_i, T_{i+1}]\) and satisfies the a priori estimates from Step 3. Furthermore, these solutions are uniformly bounded in \( X \), see Theorem 3.4, Theorem 3.5 and Remark 3.7, which eliminates any possibility of a blow-up on \([T_i, T_{i+1}]\). Thus it is admissible to take \( c_i^\pm(T_{i+1}) \) as initial value for the \( i + 1 \)-th solutions \( c_{i+1}^\pm \). This leads to a continuation to the solution on the whole time interval \([0, T]\) and consequently to a global solution.
3.4 Uniqueness

**Theorem 3.11** Any weak solution \((\Phi, v, p, c^+, c^-)\) of (1a)–(1h) according to Definition 3.3 is unique.

In the following, we denote all constants generically by \(C\) which nevertheless could change from line to line in the estimates.

**Proof.** We show uniqueness of weak solutions by using as test functions the difference of two solutions, indexed by 1 and 2 and corresponding to the same initial values and boundary data. To this end, we denote by \(\tau^\pm := c_1^\pm - c_2^\pm\), \(\Phi := \Phi_1 - \Phi_2\), \(v := v_1 - v_2\), \(p := p_1 - p_2\) the differences of the respective solutions. We now subtract the equations for \(c_1^\pm\) and \(c_2^\pm\), test with the differences \(\tau^\pm\), and arrive at

\[
\frac{1}{2} \frac{d}{dt} ||\tau^\pm||^2_{L^2(\Omega)} + \alpha_D ||\nabla \tau^\pm||^2_{L^2(\Omega)}
+ \int_\Omega (\tau_1^\pm + v_2 \tau^\pm) \cdot \nabla \tau^\pm \pm D (\nabla \Phi_1 \tau^\pm - \nabla \Phi_2 \tau^\pm) \cdot \nabla \tau^\pm \, dx
\leq \int_\Omega (\tau^+ - \tau^-) \tau^\pm \, dx.
\]

Taking the properties of the velocity field into account and using Hölder’s inequality, leads to

\[
\frac{1}{2} \frac{d}{dt} ||\tau^\pm||^2_{L^2(\Omega)} + \alpha_D ||\nabla \tau^\pm||^2_{L^2(\Omega)}
\leq ||\tau||_{L^2(\Omega)} ||c_1^\pm||_{L^\infty(\Omega)} ||\nabla \tau^\pm||_{L^2(\Omega)} + ||\nabla \Phi||_{L^2(\Omega)} ||c_1^\pm||_{L^\infty(\Omega)} ||\nabla \tau^\pm||_{L^2(\Omega)}
+ ||\nabla \Phi_2||_{L^6(\Omega)} ||\tau^\pm||_{L^4(\Omega)} ||\nabla \tau^\pm||_{L^2(\Omega)} + \int_\Omega (\tau^+ - \tau^-) \tau^\pm \, dx.
\]

Including the following stability estimates analogous to (17) and (18)

\[
||\nabla \Phi||_{L^2(\Omega)} \leq C ||\tau^+ - \tau^-||_{L^2(\Omega)},
||\tau||_{L^2(\Omega)} \leq C \left(||\tau^+ - \tau^-||_{L^2(\Omega)} ||\Phi_2||_{L^6(\Omega)} + ||c_1^+ - c_1^-||_{L^\infty(\Omega)} ||\Phi||_{L^2(\Omega)} \right),
\]

leads, together with the interpolation inequality \(||\varphi||_{L^\gamma(\Omega)} \leq C ||\varphi||_{L^2(\Omega)} ||\nabla \varphi||_{L^2(\Omega)}\), after summation and application of Young’s inequality, cf. [27], to

\[
\frac{d}{dt} \left( \sum_{\pm} ||\tau^\pm||^2_{L^2(\Omega)} \right) + \left( \sum_{\pm} ||\nabla \tau^\pm||^2_{L^2(\Omega)} \right)
\leq C \left( \max_{\pm} ||c_1^\pm||_{L^\infty(\Omega)}, ||\nabla \Phi_2||_{L^6(\Omega)} \right) \left( \sum_{\pm} ||\tau^\pm||^2_{L^2(\Omega)} \right).
\]

Gronwall’s Lemma, cf. [27], then implies

\[
\sum_{\pm} ||\tau^\pm(t)||^2_{L^2(\Omega)}
\leq \exp \left( \int_0^t C \left( \max_{\pm} ||c_1^\pm||_{L^\infty(\Omega)}, ||\nabla \Phi_2||_{L^6(\Omega)} \right) \, ds \right) \sum_{\pm} ||\tau^\pm(0)||^2_{L^2(\Omega)},
\]

which concludes the proof since \(\tau^\pm(0) = c^{\pm,0} - c^{\pm,0} = 0\).

\(\square\)
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