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Author(s): Markus Bause and Peter Knabner
Published by: Society for Industrial and Applied Mathematics
Stable URL: https://www.jstor.org/stable/4101046
Accessed: 02-05-2019 07:54 UTC

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UNIFORM ERROR ANALYSIS FOR LAGRANGE–GALERKIN APPROXIMATIONS OF CONVECTION-DOMINATED PROBLEMS*

MARKUS BAUSE† AND PETER KNABNER‡

Abstract. In this paper we present a rigorous error analysis for the Lagrange–Galerkin method applied to convection-dominated diffusion problems. We prove new error estimates in which the constants depend on norms of the data and not of the solution and do not tend to infinity in the hyperbolic limit. This is in contrast to other results in this field. For the time discretization, uniform convergence with respect to the diffusion parameter of order $O(k^{\varepsilon}t_n)$ is shown for initial values in $L^2$ and $O(k)$ for initial values in $H^2$. For the spatial discretization with linear finite elements, we verify uniform convergence of order $O(k^2 + \min\{h, h^2/k\})$ for data in $H^2$. By interpolation of Banach spaces, suboptimal convergence rates are derived under less restrictive assumptions. The analysis is heavily based on a priori estimates, uniform in the diffusion parameter, for the solution of the continuous and the semidiscrete problem. They are derived in a Lagrangian framework by transforming the Eulerian coordinates completely into subcharacteristic coordinates. Finally, we illustrate the error estimates by some numerical results.

Key words. convection-diffusion equation, Lagrange–Galerkin scheme, characteristics, finite element method, $\varepsilon$-uniform convergence

AMS subject classifications. 65M12, 65M15, 65M25, 65M60

PII. S0036142900367478

1. Introduction. In this work we present a rigorous error analysis for Lagrange–Galerkin approximations of the linear convection-diffusion problem

\begin{align}
\frac{\partial u}{\partial t} + b \cdot \nabla u - \varepsilon \Delta u &= f & \text{in } \Omega \quad &\text{for } t \in (0,T], \\
\quad u &= 0 & \text{on } \partial \Omega & \text{for } t \in (0,T], \quad u(0) = u_0 & \text{in } \Omega,
\end{align}

where $\varepsilon$ is a constant small perturbation parameter with $0 < \varepsilon \ll 1$. For $u_0 \in L^2(\Omega)$ we prove $\varepsilon$-uniform convergence of first order for the time discretization. Due to the lack of regularity of the solution the error constant becomes singular as $t \to 0$. If $u_0 \in H^2(\Omega)$, we show that the error estimate holds uniformly in time. Then, for a scale of regularity assumptions about $u_0$ and $f$, we derive some error estimates for the spatial discretization of the corresponding semidiscrete problem. Again, the error constants do not tend to infinity in the hyperbolic limit $\varepsilon \to 0$. Together, the estimates provide a rigorous $\varepsilon$-uniform error analysis for the Lagrange–Galerkin method. In (1.1), (1.2) $\Omega$ is a bounded domain in $\mathbb{R}^d$ with $d \geq 2$, and $b = b(t,x)$ is a divergence-free velocity field vanishing on the boundary $\partial \Omega$. Convection-dominated diffusion problems appear in many subjects like reactive solute transport in porous media [22], simulation of oil extraction from underground reservoirs [11], and convective heat transport with large Péclet numbers [19]. Problem (1.1), (1.2) can also be interpreted as a model for the Navier–Stokes equations; see [18]. Our analysis may be generalized to more sophisticated problems; cf. section 7. However, considering more complex models in this paper seemed to overburden the presentation.

*Received by the editors January 3, 2000; accepted for publication (in revised form) September 17, 2001; published electronically February 8, 2002.

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First, we briefly recall the construction of the Lagrange–Galerkin approach. Let $X: (t; s, x) \in [0, T] \times \Omega \mapsto X(t; s, x)$ denote the position at time $t$ of the particle which has been driven by the velocity field $b$ and is at position $x$ at time $t = s$. The parametric representation $t \mapsto X(t; s, x)$ of the trajectory may be determined from

\begin{equation}
(d/dt)X(t; s, x) = b(t, X(t; s, x)), \quad X(s; s, x) = x.
\end{equation}

If the trajectories are taken to start at $s = t_n$ and we consider the time interval $[t_{n-1}, t_n]$ on which the time derivative $\partial_t \bar{u}$ of

$$\bar{u}(t, x) := u(t, X(t; s, x))$$

is approximated by the backward Euler method, we can write the following implicit discretization scheme for (1.1), (1.2): set $u^0 := u_0$; then for $n = 1, 2, \ldots, N$ solve

\begin{equation}
k^{-1}(u^n - u^{n-1} \circ X_{n-1}) - \varepsilon \Delta u^n = f(t_n) \quad \text{in } \Omega,
\end{equation}

where $k = T/N$, $N \in \mathbb{N}$, is the time step size, $t_n = n k$ and $X_{n-1}(x) = X(t_{n-1}; t_n, x)$.

Discretizing (1.4) by the finite element method, the resulting fully discretized scheme reads as follows: given $u_0^h \in H_h$, for each $n = 1, 2, \ldots, N$ find $u^n_h \in H_h$ such that

\begin{equation}
k^{-1}(u^n_h - u^{n-1} \circ X_{n-1}, v_h) + \varepsilon \langle \nabla u^n_h, \nabla v_h \rangle = \langle f(t_n), v_h \rangle \quad \text{for all } v_h \in H_h,
\end{equation}

where $h$ is the spatial discretization parameter and $H_h$ is a conforming finite element space. The brackets $\langle \cdot, \cdot \rangle$ denote the standard inner product in $L^2(\Omega)$. In practice, the trajectories $X_{n-1}$ in (1.5) are approximated by a discretization scheme, typically the backward Euler scheme or an explicit Runge–Kutta scheme of higher order applied to (1.3). For simplicity, we assume in this work that the trajectories can be computed exactly. It seems that this simplification is not essential, and the results still hold for approximations of $X_{n-1}$. Further, we suppose that the integrals in (1.5) are evaluated exactly and not by a quadrature formula. In this way, we avoid certain stability problems related to the Lagrange–Galerkin scheme (see [26, 31]).

For the Lagrange–Galerkin scheme (1.5), Douglas and Russell [9], Süli [31], and Douglas, Huang, and Pereira [10] have proven error estimates of the form

\begin{equation}
\|u(t_n) - u^n_h\|_{L^2(\mathbb{R}^d)} \leq c(k + h^\alpha), \quad \alpha \in \{1, 2\},
\end{equation}

if linear finite elements are used. Here, $\Omega = \mathbb{R}^d$ is supposed. Pironneau [28] has shown

\begin{equation}
\|u(t_n) - u^n_h\|_{L^2(\Omega)} \leq c(k + h + h^2/k)
\end{equation}

under the assumption that the normal component of $b$ vanishes on $\partial \Omega$. The constants $c$ in (1.6) and (1.7) depend on higher order norms of the solution $u$ of (1.1), (1.2) and in [9, 10] even reciprocally on $\varepsilon$. The norms of the solution may explode in the hyperbolic limit $\varepsilon \to 0$ such that the estimates become meaningless. Thus, at least from the theoretical point of view, the convergence behavior of the Lagrange–Galerkin scheme is unclear if $\varepsilon$ is close to 0. In particular for rough initial values $u_0 \in H^s(\Omega)$ with $s < 2$, the norms arising in the error estimates of the literature do not remain finite.

In the last years there is a growing awareness of the dangers of neglecting the effect of parameter dependence; see [20, 30]. Johnson, Rannacher, and Boman [20] have observed in the case of the incompressible Navier–Stokes equations that existing
analyses often contain constants that depend on \( \exp(R) \), where \( R \) is the Reynolds number, and conclude that “in the majority of cases of interest, the existing error analysis has no meaning.”

In order to derive some \( \varepsilon \)-uniform error estimates for the scheme (1.5) involving only constants which depend on norms of the data and not of the solution, we introduce a Lagrangian framework by transforming the Eulerian coordinates in (1.1), (1.2) completely, i.e., including the Laplacian \( \Delta \), into characteristic coordinates. The transformation is done locally in each time interval \([t_{n-1}, t_n]\). These subcharacteristics are the “natural” coordinates of the Lagrange–Galerkin scheme. We prove a series of \( \varepsilon \)-uniform a priori estimates for the solution of the transformed problem thus obtained. In Eulerian coordinates one does not succeed to establish analogous \( \varepsilon \)-uniform bounds. The estimates enable us to prove that

\[
\| u(t_n) - u_h^n \|_{L^2(\Omega)} \leq c_1 k \left( \| u_0 \|_{H^2(\Omega)} + \| f \|_{C^1([0,T];L^2(\Omega))} + \| f \|_{C([0,T];H^1(\Omega))} \right) + c_2 \left( h^2 + \min\{h^2, h^2/k\} \right) \left( \| u_0 \|_{H^2(\Omega)} + \| f \|_{C([0,T];H^2(\Omega))} \right)
\]

for \( H^2 \)-regular data \( u_0 \) and \( f \). If \( f \) vanishes, then there even holds

\[
\| u(t_n) - u_h^n \|_{L^2(\Omega)} \leq c_3 (\varepsilon k + h^2 + \min\{h^2, h^2/k\}) \| u_0 \|_{H^2(\Omega)}.
\]

However, in many applications of practical interest, the data are less regular. Therefore, we show by using an interpolation argument that

\[
\| u(t_n) - u_h^n \|_{L^2(\Omega)} \leq c_4 t_n^{-1} k \left( \| u_0 \|_{L^2(\Omega)} + \| f \|_{C^1([0,T];L^2(\Omega))} + \| f \|_{C([0,T];H^1(\Omega))} \right) + c_5 \left( h^{2\theta} + \min\{h^\theta, h^{2\theta}/k^\theta\} \right) \left( \| u_0 \|_{H^{2\theta}(\Omega)} + \| f \|_{C([0,T];H^{2\theta}(\Omega))} \right).
\]

Here, \( 0 \leq \theta \leq 1 \) with \( \theta \neq 1/4 \) is a fixed parameter which solely depends on the regularity of the prescribed data \( u_0 \) and \( f \). The exact assumptions made about \( u_0 \) and \( f \), in particular about their behavior at the boundary, are formulated later on (see Theorem 6.5). However, the basic regularity requirements already result from the norms in (1.8)–(1.10). Due to the interpolation argument, if \( \theta = 1/4 \), a norm stricter than the Slobodeckij norm of \( H^{1/2}(\Omega) \) arises on the right-hand side of (1.10). Only in order to simplify the notation, we skip the case \( \theta = 1/4 \); cf. the remarks following Lemma 5.1. The estimates (1.8)–(1.10) are the main result of our \( \varepsilon \)-uniform convergence theory. We explicitly remark that \( c_1, \ldots, c_5 \) in (1.8)–(1.10) do not depend on \( \varepsilon \).

In our error analysis we separate the error of time and spatial discretization. This has also been done by Pironneau [28]. By virtue of the construction of the Lagrange–Galerkin scheme it seems to be a natural approach. We do not know how and if it is possible at all to prove the error estimates (1.8)–(1.10) without the splitting by considering temporal and spatial discretization simultaneously.

The aim of this paper is to develop the essence of the Lagrangian framework, demonstrate its power and provide some rigorous \( \varepsilon \)-uniform error estimates for the scheme (1.5). In section 7 we discuss some extensions to more sophisticated problems. Further, we make some remarks on the suboptimal term \( \min\{h^\theta, h^{2\theta}/k^\theta\} \) in (1.8)–(1.10), and to illustrate our error estimates, we present some numerical results.

2. Notations and preliminaries. We consider a bounded domain \( \Omega \subset \mathbb{R}^d \) with \( d \geq 2 \), assumed to have a Lipschitz continuous boundary and to satisfy a further condition (A1), stated below. In later sections, \( \Omega \) is assumed to be polyhedral bounded.
\(L^2(\Omega)\), or simply \(L^2\), is the space of functions defined and quadratically summable in \(\Omega\), and \(\| \cdot \| \equiv \| \cdot \|_{L^2(\Omega)}\) is its norm. \(\mathcal{L}(L^2)\) denotes the space of all bounded linear mappings from \(L^2\) into \(L^2\). Its norm is defined by

\[
\|F\| := \sup\{\|Fv\|/\|v\| \mid v \in L^2, \ v \neq 0\}, \quad F \in \mathcal{L}(L^2).
\]

Let \(m\) be a nonnegative integer. \(H^m \equiv H^m(\Omega)\) consists of all functions which, together with their derivatives in the sense of distributions up to order \(m\), belong to \(L^2\). In multi-index notation, the norm of a function \(v \in H^m\) is given by

\[
\|v\|_m := \left(\sum_{|\alpha| \leq m} \|\partial^\alpha v\|^2\right)^{1/2}.
\]

For nonintegral values \(\alpha > 0\), \(\alpha = \lfloor \alpha \rfloor + \lambda\) with \(0 < \lambda < 1\), let the space \(H^\alpha \equiv H^\alpha(\Omega)\) be defined according to the recipe given by Slobodeckij (see [35, p. 61]) through

\[
H^\alpha = \{v \in L^2 \mid D^m v \in L^2 \text{ for } |m| \leq \lfloor \alpha \rfloor \text{ and } I_\lambda(D^m v) < \infty\},
\]

where \(I_\lambda(\varphi) := \int_\Omega \int_\Omega \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{d+2\lambda}} \, dx \, dy\). We equip \(H^\alpha\) with the norm

\[
\|v\|_\alpha := \left(\|v\|^2_{[\alpha]} + \sum_{|m| \leq \lfloor \alpha \rfloor} I_\lambda(D^m v)^2\right)^{1/2}.
\]

\(W^{1,\infty} \equiv W^{1,\infty}(\Omega)\) consists of all functions which, together with their first order derivatives in the sense of distributions, are essentially bounded and measurable in \(\Omega\). The norm in \(W^{1,\infty}\) is defined by

\[
\|v\|_{W^{1,\infty}} := \max\{\text{ess sup}_{x \in \Omega}\|\partial^\alpha v(x)\| \mid |\alpha| \leq 1\}.
\]

\(C^m(\Omega)\) is the space of functions \(m\)-times continuously differentiable in \(\Omega\). Among them, the collection of functions whose supports are compact in \(\Omega\) will be denoted by \(C^m_0(\Omega)\). The closure of \(C^m_0(\Omega)\) with respect to the norm of \(H^\alpha\), \(\alpha \in \mathbb{R}^+\), is denoted by \(H^\alpha_0\). \(H^{-1} \equiv H^{-1}(\Omega)\) is the dual space of \(H^1_0\). If \(v \in H^{-1}\), we define the norm

\[
\|v\|_{-1} := \sup\{\langle v, \varphi \rangle/\|\nabla \varphi\| \mid \varphi \in H^1_0\}.
\]

Another set of elements of \(C^m(\Omega)\) is \(C^m(\overline{\Omega})\). It consists of all functions \(v\) for which \(\partial^\alpha v\) can be extended continuously to \(\Omega\) for \(|\alpha| \leq m\). The norm in \(C^m(\overline{\Omega})\) is

\[
\|v\|_{C^m(\overline{\Omega})} := \max\{|\partial^\alpha v(x)| \mid |\alpha| \leq m, \ x \in \overline{\Omega}\}.
\]

Let \(J \subset \mathbb{R}\) be an interval. Then, \(C^m(J \times \overline{\Omega})\) is defined analogously. Further let \(B\) be a Banach space. By \(C^m(J; B)\), or simply \(C(J; B)\) if \(m = 0\), we denote the space of all functions \(v(t)\) from the interval \(J\) into \(B\) that have \(m\) continuous derivatives in \(J\). If \(J\) is compact, we equip \(C^m(J; B)\) with the norm

\[
\|v\|_{C^m(J; B)} := \max\{\|\partial^l v(t)\|_B \mid t \in J, \ 0 \leq l \leq m\}.
\]

As usual, for a space-time function \(v: (t, x) \mapsto v(t, x)\), we use the abbreviation \(v(t)\) for the function \(v(t): x \mapsto (v(t))(x) = v(t, x)\) defined in \(\Omega\). In the notation we make no distinction between \(\mathbb{R}\)-valued functions and vector- or matrix-valued functions as well as between their function spaces. From now on we set \(J := [0, T]\).

As mentioned above, we need an additional assumption about \(\Omega\):
Let $D = D(x)$ be a real $d \times d$ matrix with components $d_{ij} \in W^{1,\infty}$, $d_{ij}(x) = d_{ji}(x)$, and $\zeta^T D(x) \zeta \geq \delta |\zeta|^2$, $\delta > 0$, for every $x \in \Omega$ and all $\zeta \in \mathbb{R}^d$. The unique solution $v \in H_0^1$ of the steady problem

\begin{equation}
-\nabla \cdot (D \nabla v) = g \text{ in } \Omega, \quad v|_{\partial \Omega} = 0,
\end{equation}

for prescribed $g \in L^2$, satisfies

\begin{equation}
\|v\| \leq A \|g\|,
\end{equation}

where $A = A(d, \Omega, \delta, M)$ depends on $d$, $\Omega$, $\delta$, $M := \max\{|d_{ij}|_{W^{1,\infty}} | 1 \leq i, j \leq d\}$, and $A(d, \Omega, \delta, \cdot)$ is a monotonically increasing function on $\mathbb{R}^+$. In what follows, we drop $d$ and $\Omega$ in the argument list of $A$ and write simply $A(\delta, M)$. If $\partial \Omega$ is of class $C^2$, assumption (A1) holds with $A = c (1 + M^2)^2$, where $c$ does not depend on $D$; see [5, p. 24]. Moreover, if $\Omega$ is a convex subset of $\mathbb{R}^d$, assumption (A1) is satisfied for all matrices with Lipschitz-continuous components on $\Omega$. This follows from combining [14, Theorem 3.1.3.1] with [14, Theorem 3.2.1.2] and analyzing the dependence on $D$ of those constants which arise in the proofs.

Next, we define assumptions about the prescribed data for problem (1.1), (1.2):

(A2). The initial value $u_0 = u_0(x)$ and the right-hand side $f = f(t, x)$ satisfy

$u_0 \in L^2$ and $f \in C^1(J; L^2) \cap C(J; H')$.

Some of the results which we present below hold still under weaker assumptions about $f$. However, to prove (1.8)-(1.10) we need condition (A2). Therefore, to simplify the notation, we suppose throughout the paper that the conditions (A1) and (A2) are satisfied. To establish $\varepsilon$-uniform convergence for the spatial discretization, we need an additional assumption about $u_0$ and $f$ which will be formulated later on in the appropriate theorems.

We assume that $b \in C(J; C^2(\Omega))$. Moreover, for the sake of simplicity we suppose that $b$ is divergence-free for $(t, x) \in J \times \Omega$ and vanishes on $\partial \Omega$ for $t \in J$. As in other publications in this field (see [9, 28, 31, 10]), we thus do not consider inflow and outflow boundaries. The theoretical analysis of Lagrange–Galerkin schemes with inflow and outflow boundaries is still an open problem. Although the latter assumptions about $b$ are of some importance in our analysis, we believe that they are not essential and our techniques are extendible to the more general situations; see section 7.

For given $t_n \in [0, T]$ and $x \in \Omega$, let $X^n : t \mapsto X^n(t) := X(t; t_n, x)$ be the unique solution of the system (1.3) of ordinary differential equations. The condition $b \in C(J; C^2(\Omega))$ ensures that $X^n \in C(J; C^2(\Omega))$; see, e.g., [15, p. 100]. Since $b \equiv 0$ on $\partial \Omega$ by assumption, the trajectories always remain in $\Omega$. In addition, from Liouville’s differential equation (see [8, p. 11]) we obtain that $\det \nabla X^n(t) = 1$. This implies

\begin{equation}
\langle f, g \rangle = \langle f \circ X^n(t), g \circ X^n(t) \rangle
\end{equation}

for any function $f, g \in L^2$ and, moreover, for $f \in L^2$ the identity

\begin{equation}
\|f\| = \|f \circ X^n(t)\|.
\end{equation}

We define the operator $E^n : L^2 \mapsto L^2$, $n = 1, \ldots, N$, through

\begin{equation}
E^n \phi := \phi \circ X^n(t_{n-1}), \quad \phi \in L^2.
\end{equation}
The transformation of the Laplacian $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ with respect to a $C^2$-diffeomorphism $\Phi: \Omega \mapsto \Omega, \, x \mapsto \xi = \Phi(x)$ is defined by the formula

$$\Delta^\Phi(u \circ \Phi) = (\Delta u) \circ \Phi$$

for all $u \in C^2(\Omega)$. The operator $\Delta^\Phi$ can be represented as follows.

**Lemma 2.1.** Let $\Phi: \Omega \mapsto \Omega, \, x \mapsto \xi = \Phi(x)$ be a $C^2$-diffeomorphism on $\Omega$ with $\det \nabla \Phi = 1$. Then we have

$$\Delta^\Phi = \sum_{i,j=1}^{d} \partial_{x_i}(g_{ij} \partial_{x_j}), \quad G := (\nabla \Phi)^{-1} (\nabla \Phi)^{-T}.$$

Here, the functions $g_{ij}$ denote the components of the $d \times d$ matrix $G(x)$ and $(\nabla \Phi)(x)$ is the functional matrix of $\Phi$ with components $(\nabla \Phi)_{i,j} = \frac{\partial \Phi_i}{\partial x_j}$.

**Proof.** Let $u \in C^2(\Omega), \, v \in C^0_0(\Omega)$ and $\tilde{u} := u \circ \Phi, \, \tilde{v} := v \circ \Phi$. Since $\det \nabla \Phi = 1$ by assumption, the substitution rule gives

$$\int_{\Omega} (\Delta u) v \, d\xi = \int_{\Omega} ((\Delta u) \circ \Phi)(v \circ \Phi) \, dx = \int_{\Omega} (\Delta^\Phi \tilde{u}) \tilde{v} \, dx. \tag{2.6}$$

On the other hand, integration by parts yields

$$\int_{\Omega} (\Delta u) v \, d\xi = -\int_{\Omega} \nabla u \cdot \nabla v \, d\xi = -\int_{\Omega} ((\nabla u) \circ \Phi) \cdot ((\nabla v) \circ \Phi) \, dx. \tag{2.7}$$

The chain rule implies

$$\nabla((\psi \circ \Phi) = (\nabla \psi)^T ((\nabla \psi) \circ \Phi)$$

for $\psi = u, v$. Substituting the expression thus obtained for $(\nabla u) \circ \Phi$ and $(\nabla v) \circ \Phi$ in the right-hand side of (2.7) and integrating subsequently by parts, we get

$$\int_{\Omega} ((\nabla u) \circ \Phi) \cdot ((\nabla v) \circ \Phi) \, dx = \int_{\Omega} \nabla \cdot ((\nabla \Phi)^{-1} (\nabla \Phi)^{-T} \nabla \tilde{u}) \tilde{v} \, dx. \tag{2.8}$$

Finally, combining (2.6), (2.7), and (2.8) gives

$$\int_{\Omega} (\Delta^\Phi \tilde{u}) \tilde{v} \, dx = \int_{\Omega} \nabla \cdot (G \nabla \tilde{u}) \tilde{v} \, dx, \quad G = (\nabla \Phi)^{-1} (\nabla \Phi)^{-T}. \tag{2.9}$$

Since (2.9) holds for all $v \in C^0_0(\Omega)$, we deduce that $\Delta^\Phi \tilde{u} = \nabla \cdot (G \nabla \tilde{u})$. \quad \Box

Next, we define the abbreviations

$$\tilde{u}^n(t, x) := u(t, X^n(t)) = u(t, X(t; t_n, x)), \tag{2.10}$$

$$D^n(t, x) := (\nabla X^n(t))^{-1} \varepsilon I (\nabla X^n(t))^{-T} = (\nabla X(t; t_n, x))^{-1} \varepsilon I (\nabla X(t; t_n, x))^{-T}$$

for $t \in J_n := [t_{n-1}, t_n], \, x \in \bar{\Omega}$, and $n = 1, \ldots, N$, where $I$ is the identity matrix and $(\nabla X^n)_{i,j} = \frac{\partial X^n_i}{\partial x_j}$. For $n = 0$ we set $\tilde{u}^0(t_0, x) := u(t_0, x)$. Obviously, $D^n = D^n(t, x)$ is symmetric. The assumptions about $b$ ensure that the matrix elements $d^n_{ij}$ satisfy $d^n_{ij} \in C^1(J_n \times \bar{\Omega})$ and $\partial_t d^n_{ij} \in C(J_n; C^1(\bar{\Omega}))$; see, e.g., [15, p. 100]. Moreover, there exist constants $B_l = B_l(d, \bar{\Omega}, b, T), \, l = 0, \ldots, 6$, independent of $\varepsilon$, such that

$$\|b_m\|_{C(J_n \times \bar{\Omega})} \leq B_0, \quad \|\partial_{x_i}X^n_m\|_{C(J_n \times \bar{\Omega})} \leq B_1,$$

$$\|\partial_{x_i}X^n_m\|_{C(J_n \times \bar{\Omega})} \leq B_2, \quad \max_{t \in J_n} \|d^n_{ij}(t)\|_{C^1(\bar{\Omega})} \leq \varepsilon B_3,$$

$$\max_{t \in J_n} \|\partial_t d^n_{ij}(t)\|_{C^1(\bar{\Omega})} \leq \varepsilon B_4, \quad \|\partial^p d^n_{ij}\|_{C(J_n \times \bar{\Omega})} \leq \varepsilon B_{5+p}.$$
with $1 \leq i, j, m \leq d$ and $p = 0, 1$, for any $n = 1, \ldots, N$. We are primarily not interested in the dependence of the a priori estimates on $b$. Therefore, we do not provide more detailed estimates for the norms in (2.11). $D^n$ is uniformly positive definite.

**Lemma 2.2.** For $n = 1, \ldots, N$ and $(t, x) \in J_n \times \overline{\Omega}$, the matrix $D^n(t, x)$ satisfies

\begin{equation}
(2.12) \quad \zeta^T D^n(t, x) \zeta \geq d^{-2} B_1^{-2} \varepsilon |\zeta|^2, \quad \zeta \in \mathbb{R}^d.
\end{equation}

**Proof.** For $\zeta \in \mathbb{R}^d$ and $(t, x) \in J_n \times \overline{\Omega}$ we put $\eta := \left(\nabla X^n(t)\right)^{-T} \zeta$. It follows that

\begin{equation}
(2.13) \quad \zeta^T D^n(t, x) \zeta = \zeta^T \left(\nabla X^n(t)\right)^{-1} \varepsilon I \left(\nabla X^n(t)\right)^{-T} \zeta = \varepsilon |\eta|^2.
\end{equation}

On the other hand, we find

\begin{equation}
(2.14) \quad |\zeta|^2 = |\left(\nabla X^n(t)\right)^T \eta|^2 \leq d^2 \max\{|\partial_{x_i} X^n_m(t)| \mid 1 \leq i, m \leq d\} |\eta|^2 \leq d^2 B_1^2 |\eta|^2.
\end{equation}

Combining (2.13) with (2.14) gives (2.2). \qed

Finally, we introduce the notation

\[ a^n_\psi(\psi, \chi) := \langle D^n(t) \nabla \psi, \nabla \chi \rangle, \quad a^0_\psi(\psi, \chi) := \langle \varepsilon \nabla \psi, \nabla \chi \rangle, \quad a^n_t(\psi, \chi) := \langle d/dt a^n_\psi(\psi, \chi) \rangle \]

for $\psi, \chi \in H^1_0$, $n = 1, \ldots, N$. The two $m$-sectorial operators (see [21, p. 280]) $A^n(t)$ and $A \equiv A^0(t_0)$ in $L^2$ can now be defined through the relations

\begin{equation}
(2.15) \quad \langle A^n(t) \psi, \chi \rangle = a^n_\psi(\psi, \chi), \quad \langle A \psi, \chi \rangle = a^0_\psi(\psi, \chi),
\end{equation}

respectively, for $\psi \in D(A^n(t)) \subset H^1_0$, $\psi \in D(A)$, respectively, and $\chi \in H^1_0$. Assumption (A1) implies that $D(A^n(t)) = H^2 \cap H^1_0$ and $D(A) = H^2 \cap H^1_0$ and, moreover,

\begin{equation}
(2.16) \quad \|\psi\|_2 \leq \varepsilon^{-1} A_b \|A^n(t) \psi\|, \quad A_b := A(d^{-2} B_1^{-2}, B_3)
\end{equation}

for $\psi \in D(A^n(t))$. Thus, the domain of $A^n(t)$ is independent of $t$, i.e., $D(A^n(t)) = D(A^n(t_{n-1}))$ for $t \in [t_{n-1}, t_n]$. Inequality (2.16) yields

\begin{equation}
(2.17) \quad \left\| (A^n(t) - A^n(\tau)) A^n(s)^{-1} \psi \right\| \leq \varepsilon 2 d^2 B_4 |t - \tau| \|A^n(s)^{-1} \psi\|_2 \leq 2 d^2 B_4 A_b |t - \tau| \|A^n(s) A^n(s)^{-1} \psi\| = 2 d^2 B_4 A_b |t - \tau| \|\psi\|
\end{equation}

for any $t, s, \tau \in J_n$ and $\psi \in L^2$. Hence, $-A^n(t)$ generates a uniquely determined operator-valued function $U^n : \{(t, s) \mid (t, s) \in J_n \times J_n, \ s \leq t\}$ with the following properties: $U^n(t, s)$ is strongly continuous, $U^n(t, s)$ is strongly continuous differentiable on $t > s$ with respect to $t$, $U^n(t, s) D(A^n(t_{n-1})) \subset D(A^n(t_{n-1}))$ and

\begin{equation}
(\partial/\partial t)(U^n(t, s)) \psi + A^n(t) U^n(t, s) \psi = 0, \quad U^n(t, t) = I,
\end{equation}

where $I$ is the identity operator. For $t \in (t_{n-1}, t_n]$ and $\psi \in D(A^n(t_{n-1}))$, the function $U^n(t, s) \psi$ is differentiable with respect to $s$ in $t_{n-1} \leq s \leq t$ and satisfies

\begin{equation}
(2.18) \quad (\partial/\partial s)(U^n(t, s)) \psi = U^n(t, s) A^n(s) \psi.
\end{equation}

Further, it is a standard observation (cf. [2, p. 11]) that

\begin{equation}
(2.19) \quad \|U^n(t, s)\| \leq 1, \quad t, s \in J_n.
\end{equation}

A proof of these facts can be found in, e.g., [13, 32].
3. A priori analysis in Lagrangian coordinates. In this section we first rewrite the convection-diffusion problem (1.1), (1.2) in Lagrangian coordinates. This is done locally in time, which means that we transform the Eulerian coordinates into the subcharacteristic coordinates \( X(t; t_n, x) \) with \( t \in [t_{n-1}, t_n] \) and do not use global characteristic lines. For the solution of the transformed problem, we then derive a hierarchical sequence of a priori estimates which hold uniformly with respect to the parameter \( \varepsilon \) in the sense that the bounds on the right-hand side of the estimates do not increase for \( \varepsilon \) tending to zero. We explicitly point out that this \( \varepsilon \)-uniformity is the key issue of our a priori analysis. The error estimates below will then depend heavily on these a priori estimates.

**Theorem 3.1.** Suppose conditions (A1) and (A2) are satisfied. Then problem (1.1), (1.2) has a unique solution \( u \in C(J; L^2) \cap C^1((0, T]; L^2) \cap C((0, T]; H^2 \cap H^0_0) \). Now assume \( u_0 \in H^2 \cap H^0_0 \). Then, for \( n = 1, \ldots, N \) the function \( \bar{u}^n = u \circ X^n \) satisfies

\[
\bar{u}^n \in C(J_n; H^2 \cap H^0_0) \cap C^1(J_n; L^2).
\]

Moreover, \( \bar{u}^n \) is the unique solution of the initial boundary value problem

\[
\left\{ \begin{array}{ll}
\partial_t \bar{u}^n - \nabla \cdot (D^n \nabla \bar{u}^n) &= \tilde{f}^n \quad \text{in } \Omega, \\
\bar{u}^n &= 0 \quad \text{on } \partial \Omega, \\
\bar{u}^n(t_{n-1}) &= \bar{u}^{n-1}(t_{n-1}) \circ X^n(t_{n-1}) \quad \text{in } \Omega,
\end{array} \right.
\]

and is expressible as

\[
\bar{u}^n(t) = U^n(t, t_{n-1}) \bar{u}^n(t_{n-1}) + \int_{t_{n-1}}^{t} U^n(t, s) \tilde{f}^n(s) \, ds, \quad t \in J_n.
\]

Under the weaker assumption \( u_0 \in H^1_0 \), (3.1)–(3.4) hold with (3.1) replaced by

\[
\bar{u}^1 \in C(J_1; L^2) \cap C((0, t_1]; H^2 \cap H^0_0) \cap C((0, t_1]; L^2)
\]

for \( n = 1 \). Under no further assumption in addition to (A1) and (A2), (3.1)–(3.4) are satisfied with (3.1) replaced by

\[
\bar{u}^1 \in C(J_1; L^2) \cap C((0, t_1]; H^2 \cap H^0_0) \cap C((0, t_1]; L^2)
\]

in the case \( n = 1 \).

**Proof.** The first assertion is a standard result for parabolic evolution equations; see [13, 32]. Suppose \( u_0 \in H^2 \cap H^0_0 \). Let \( \bar{u}^n = u \circ X^n \). By (2.4) and (2.11) we get

\[
\| \bar{u}^n(t) \|^2 = \| u(t, X^n(t)) \|^2 + \| \nabla u(t, X^n(t)) \|^2 + \sum_{|\alpha|=2} \| \partial^\alpha u(t, X^n(t)) \|^2 \leq C_0 \| u \|_{C(J; H^2)}^2
\]

with \( C_0 := \max \{1, 2d^2 B_2^2 + 2d^3 B_2^3, 2d^4 B_1^4 \} \). Hence, there holds \( \bar{u}^n(t) \in H^2 \) for \( t \in J_n \). Since the velocity field \( b \) vanishes on \( \partial \Omega \) by assumption, we have \( \bar{u}^n(t) \in H^0_0 \) and thus the boundary condition in (3.3). The chain rule implies

\[
\partial_t \bar{u}^n(t) = (\partial_t u(t) + b(t) \cdot \nabla u(t))(t, X^n(t)),
\]

which together with identity (2.4) shows

\[
\| \partial_t \bar{u}^n(t) \| = \| \partial_t u(t) + b(t) \cdot \nabla u(t) \| \leq \| \partial_t u \|_{C(J; L^2)} + \| D^{1/2} B_0 u \|_{C(J; H^1)},
\]

and thus \( \partial_t \bar{u}^n(t) \in L^2 \) for \( t \in J_n \). The differential equation (3.2) is now an obvious consequence of (3.5) and Lemma 2.1. Recalling \( X^{n-1}(t_{n-1}) = x \), the initial condition in (3.3) follows directly from the definition of \( \bar{u}^n \) and \( \bar{u}^{n-1} \).
Next, we assume \( t_{n-1} < s < t \leq t_n \). Then, from (2.18) and (3.2) it follows that
\[
\partial_s (U^n(t, s)\tilde{u}^n(s)) = \tilde{u}^n(t, s)\partial_s \tilde{u}^n(s) + U^n(t, s)A^n(s)\tilde{u}^n(s) = U^n(t, s)\tilde{f}^n(s).
\]
Integrating this equation from \( t_{n-1} \) to \( t \), we obtain (3.4). It is well known that (3.4) is the unique solution of (3.2), (3.3). Moreover, there holds \( \partial_t \tilde{u}^n \in C(J_n; L^2) \) and \( A^n \tilde{u}^n \in C(J_n; L^2) \). A proof of these facts can be found in [13, 32]. We note that condition (A2) ensures that \( \tilde{f}^n = f \circ X^n \in C^0(J_n; L^2) \). Finally, by (2.16) we have
\[
\|\tilde{u}^n(t) - \tilde{u}^n(s)\|_2 \leq \varepsilon^{-1}A_b\|A^n(t)\tilde{u}^n(t) - A^n(t)\tilde{u}^n(s)\| \leq A_b(\varepsilon^{-1}\|A^n(t)\tilde{u}^n(t) - A^n(t)\tilde{u}^n(s)\| + 2 d^2 B_4 C_0^{1/2} |t - s| \|u\|_{C(J; H^2)}).
\]
Combined with \( A^n \tilde{u}^n \in C(J_n; L^2) \), this estimate proves the continuity of \( \tilde{u}^n \) in \( H^2 \) for \( t \in J_n \) and thus property (3.1).

Under the weaker condition \( u_0 \in H^1_0 \), the assertions can be shown in almost exactly the same way as before. The proof changes only for \( n = 1 \). If \( n = 1 \), then the theory of parabolic evolution equations implies \( u \in C(J; H^1_0) \cap C^1((0, T]; L^2) \cap C((0, T]; H^2) \). This ensures that \( \tilde{u}^1(t) \in H^2_0 \) for \( t \in J_1 \) as well as \( \partial_t \tilde{u}^1(t) \in L^2 \) and \( \tilde{u}^1 \in H^2 \) for \( t \in (0, t_1] \). Moreover, the differential equation (3.2) and the variation of constants formula (3.4) hold, and well-known results for evolution equations of parabolic type show that \( \tilde{u}^1 \) satisfies \( \tilde{u}^1 \in C(J_1; L^2) \cap C^1((0, t_1]; L^2) \), \( A^1 \tilde{u}^1 \in C((0, t_1]; L^2) \) and \( (A^1)^{1/2} \tilde{u}^1 \in C(J_1; L^2) \). The continuity of \( \tilde{u}^1 \) in \( H^1 \) for \( t \in J_1 \) can now be proven similarly to (3.6) by estimating the \( H^1 \)-norm in terms of \( (A)_{1/2} \).

If only \( u_0 \in L^2 \), the first assertion of Theorem 3.1 implies \( \tilde{u}^1(t) \in L^2 \) for \( t \in J_1 \) and \( \partial_t \tilde{u}^1(t) \in L^2 \) and \( \tilde{u}^1(t) \in H^2 \cap H^0_0 \) for \( t \in (0, t_1] \). Hence, the differential equation (3.2) and formula (3.4) still hold. Finally, from (3.4) it follows that \( \tilde{u}^1 \in C(J_1; L^2) \); see [13, 32]. For \( n = 2, \ldots, N \) the proof remains unchanged. \( \square \)

Now we derive a sequence of a priori estimates for the solution of (1.1), (1.2) rewritten in subcharacteristic coordinates. We use a combination of energy and semigroup methods. This yields the sharpest estimates in the sense that the assumptions about the velocity field \( b \), primarily, and the time \( T \), secondarily, are the weakest. For example, in order to permit an extension of the Lagrangian framework to the Navier–Stokes equations whose solution admits only a limited regularity due to nonlocal compatibility conditions at \( t = 0 \) (see, e.g., [6, 16]), it is important to ensure that the a priori estimates hold under assumptions about \( b \) which are as weak as possible.

**Lemma 3.2.** Suppose conditions (A1) and (A2) are satisfied. For \( n = 1, \ldots, N \) let \( \tilde{u}^n \) be the solution of (3.2), (3.3) according to Theorem 3.1. Then there holds
\[
\|\tilde{u}^n\|_{C(J_n; L^2)} \leq \|u_0\| + T\|f\|_{C(J; L^2)}.
\]
**Proof.** By (3.4), the initial condition in (3.3) and definition (2.5) we obtain
\[
\tilde{u}^n(t) = U^n(t, t_{n-1})E^n\tilde{u}^{n-1}(t_{n-1}) + \int_{t_{n-1}}^t U^n(t, s)\tilde{f}^n(s) \, ds \\
= U^n(t, t_{n-1})E^nU^{n-1}(t_{n-1}, t_{n-2})E^{n-1}\tilde{u}^{n-2}(t_{n-2}) \\
+ U^n(t, t_{n-1})E^n\int_{t_{n-1}}^t U^{n-1}(t_{n-1}, s)\tilde{f}^{n-1}(s) \, ds + \int_{t_{n-1}}^t U^n(t, s)\tilde{f}^n(s) \, ds.
\]
Repeating this argument, i.e., using (3.4), (3.3), and (2.5), we find

\[
\tilde{u}^n(t) = U^n(t, t_{n-1}) E^n U^{n-1} E^{n-1} \ldots U^1 E^1 u_0 + \sum_{j=1}^{n-1} U^n(t, t_{n-1}) E^n U^{n-1} E^{n-1} \ldots U^j E^j f_t(s) + \int_{t_{j-1}}^{t_j} U^j(t_j, s) \tilde{f}^j(s) \, ds.
\]

In (3.8) we use the abbreviation \( U^j = U^j(t_j, t_{j-1}) \) for \( j = 1, \ldots, n - 1 \). Further, for \( j = n - 1 \) we let \( U^n(t, t_{n-1}) E^n U^{n-1} E^{n-1} \ldots U^{j+1} E^{j+1} = U^n(t, t_{n-1}) E^n \). By (2.19), (2.5), and (2.4) we now deduce that

\[
\|\tilde{u}^n(t)\| \leq \|u_0\| + \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \|\tilde{f}^j(s)\| \, ds \leq \|u_0\| + T \|f\|_{C(J; L^2)}.
\]

This completes the proof. \( \square \)

**Lemma 3.3.** Suppose conditions (A1) and (A2) are satisfied. For \( n = 1, \ldots, N \) let \( \tilde{u}^n \) be the solution of (3.2), (3.3) according to Theorem 3.1. Then there holds

\[
\sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \|\nabla \tilde{u}^j(t)\|^2 \, dt \leq \frac{d^2}{d^2 B^2_1} M_1 \left( \|u_0\|^2 + \|f\|^2_{C(J; L^2)} \right),
\]

where \( M_1 := M_2 + 2T \max\{1, T^2\}, \) \( M_2 := \max\{1, T\} \), and \( B_1 \) is defined in (2.11).

*Proof.* The differential equation (3.2) is equivalent to its weak form, namely

\[
\langle \partial_t \tilde{u}^j, v \rangle + a_t^j(\tilde{u}^j, v) = \langle \tilde{f}^j, v \rangle, \quad v \in H^1_0.
\]

Putting \( v = \tilde{u}^j \) in (3.10), by Lemma 2.12 and Cauchy–Young’s inequality we have

\[
\frac{1}{2} \frac{d}{dt} \|\tilde{u}^j\|^2 + \frac{\varepsilon}{2d^2 B^2_1} \|\nabla \tilde{u}^j\|^2 \leq \frac{1}{2} \|\tilde{f}^j\|^2 + \frac{1}{2} \|\tilde{u}^j\|^2.
\]

Now, we integrate (3.11) from \( t_{j-1} \) to \( t_j \) and sum up the resulting inequality. Since \( \|\tilde{u}^j(t_{j-1})\| = \|\tilde{u}^{j-1}(t_{j-1})\| \) by virtue of the initial condition in (3.3) and relation (2.4), we obtain

\[
\frac{1}{d^2 B^2_1} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \|\nabla \tilde{u}^j(t)\|^2 \, dt \leq \|u_0\|^2 + T \|f\|^2_{C(J; L^2)} + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \|\tilde{u}^j\|^2 \, dt.
\]

The desired estimate (3.9) follows from (3.12) and Lemma 3.2. \( \square \)

**Lemma 3.4.** Suppose conditions (A1) and (A2) are satisfied. For \( n = 1, \ldots, N \) let \( \tilde{u}^n \) be the solution of (3.2), (3.3) according to Theorem 3.1. Then there holds

\[
\sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} t \|\partial_t \tilde{u}^j(t)\|^2 \, dt \leq C_1 \left( \|u_0\|^2 + \|f\|^2_{C(J; L^2)} \right),
\]

where \( C_1 := 2 \max\{T^2, d^2 B^2_1 M_1 (B_5 + B_6 T)\} \) with \( B_i \) and \( M_1 \) being defined in (2.11) and Lemma 3.3.

*Proof.* Putting \( \phi = \partial_t \tilde{u}^j \) in (3.10), we have

\[
\|\partial_t \tilde{u}^j\|^2 + a_t^j(\tilde{u}^j, \partial_t \tilde{u}^j) = \langle \tilde{f}^j, \partial_t \tilde{u}^j \rangle.
\]
Observing
\[ 2a_t^j(\tilde{u}^j, \partial_t \tilde{u}^j) = (d/dt)a_t^j(\tilde{u}^j, \tilde{u}^j) - \tilde{a}_t^j(\tilde{u}^j, \tilde{u}^j), \]
we get
\[ 2a_t^j(\tilde{u}^j, \partial_t \tilde{u}^j) = (f_j^t, \partial_t \tilde{u}^j) + \frac{1}{2}\tilde{a}_t^j(\tilde{u}^j, \tilde{u}^j) \]
\[ \leq \frac{1}{2}\|f_j^t\|^2 + \frac{1}{2}\|\partial_t \tilde{u}^j\|^2 + \frac{1}{2}d_B\epsilon\|\nabla \tilde{u}^j\|^2. \]

Next, we multiply (3.14) with \( t \) on both sides and integrate the resulting inequality from \( t_{j-1} \) to \( t_j \). This gives
\[ \int_{t_{j-1}}^{t_j} t \|\partial_t \tilde{u}^j\|^2 dt + t_a a_t^j(\tilde{u}^j(t_j), \tilde{u}^j(t_j)) - t_{j-1} a_t^j(\tilde{u}^j(t_{j-1}), \tilde{u}^j(t_{j-1})) \]
\[ \leq \int_{t_{j-1}}^{t_j} t \|f_j^t\|^2 dt + d_B\epsilon \int_{t_{j-1}}^{t_j} t \|\nabla \tilde{u}^j\|^2 dt + d_B\epsilon \int_{t_{j-1}}^{t_j} \|\nabla \tilde{u}^j\|^2 dt. \]

The chain rule implies
\[ \nabla \tilde{u}^j(t) = \nabla (u(t) \circ X^j(t)) = (\nabla X^j(t))^T (\nabla u(t)) \circ X^j(t), \]
such that by property (2.3) it follows that
\[ a_t^j(\tilde{u}^j(t_j), \tilde{u}^j(t_j)) = (D^j(t_j) \nabla \tilde{u}^j(t_j), \nabla \tilde{u}^j(t_j)) \]
\[ = (\epsilon(\nabla u(t_{j-1})) \circ X^j(t_{j-1}), (\nabla u(t_{j-1})) \circ X^j(t_{j-1})) \]
\[ = (\epsilon \nabla u(t_{j-1}), \nabla u(t_{j-1})) = a_t^{-1}(\tilde{u}^{j-1}(t_{j-1}), \tilde{u}^{j-1}(t_{j-1})). \]

Therefore, recalling (2.4), we find
\[ \int_{t_{j-1}}^{t_j} t \|\partial_t \tilde{u}^j\|^2 dt + t_a a_t^j(\tilde{u}^j(t_j), \tilde{u}^j(t_j)) - t_{j-1} a_t^j(\tilde{u}^{j-1}(t_{j-1}), \tilde{u}^{j-1}(t_{j-1})) \]
\[ \leq kT\|f\|^2_{C(J, L^2)} + d(B_5 + B_6T) \int_{t_{j-1}}^{t_j} \epsilon \|\nabla \tilde{u}^j\|^2 dt. \]

Summing up (3.16) and combining the resulting inequality with Lemma 3.3 yields the desired estimate (3.13). □

**Lemma 3.5.** Suppose conditions (A1) and (A2) are satisfied. For \( n = 1, \ldots, N \) let \( \tilde{u}^n \) be the solution of (3.2), (3.3) according to Theorem 3.1. Then there holds
\[ \sum_{j=1}^{n} \epsilon \int_{t_{j-1}}^{t_j} t^2 \|\nabla \partial_t \tilde{u}^j(t)\|^2 dt \leq C_2 (\|u_0\|^2 + \|f\|^2_{C(J, L^2)} + \|f\|^2_{C(J, H^1)}), \]
where \( C_2 := 3d^2B^2 \max \{d^6B_4^2B_6^2M_1T^2, T^3, d^2B_4^2T^3, C_1(T + 2)\} \) with \( B_i, M_1 \) and \( C_1 \) being defined in (2.11) and Lemmas 3.3 and 3.4.

**Proof.** Differentiating with respect to \( t \) in (3.10) results in the equation
\[ \langle \partial_t^2 \tilde{u}^j, \phi \rangle + a_t^j(\partial_t \tilde{u}^j, \phi) = -\tilde{a}_t^j(\tilde{u}^j, \phi) + \langle \partial_t \tilde{f}^j, \phi \rangle, \quad \phi \in H^1_0. \]

Putting \( \phi = \partial_t \tilde{u}^j \) in (3.18), we find by (2.12) and Cauchy–Young’s inequality that
\[ \frac{1}{2}\frac{d}{dt} \|\partial_t \tilde{u}^j\|^2 + \frac{\epsilon}{2d^2B^2} \|\nabla \partial_t \tilde{u}^j\|^2 \leq -\tilde{a}_t^j(\tilde{u}^j, \partial_t \tilde{u}^j) + \langle \partial_t \tilde{f}^j, \partial_t \tilde{u}^j \rangle \]
\[ \leq \frac{d^4B_4^2B_6^2}{2} \|\nabla \tilde{u}^j\|^2 + \frac{\epsilon}{2d^2B^2} \|\nabla \partial_t \tilde{u}^j\|^2 + \frac{1}{2}\|\partial_t \tilde{f}^j\|^2 + \frac{1}{2}\|\partial_t \tilde{u}^j\|^2. \]
and thus
\[ d^2 B_1^2 \frac{d}{dt} \| \partial_t \tilde{u}^j \|^2 + \varepsilon \| \nabla \partial_t \tilde{u}^j \|^2 \leq d^6 B_1^4 B_0^2 \varepsilon \| \nabla \tilde{u}^j \|^2 + d^2 B_1^2 \| \partial_t \tilde{f}^j \|^2 + d^2 B_1^2 \| \partial_t \tilde{u}^j \|^2. \]

We multiply this relation with $t^2$ on both sides and integrate from $t_{j-1}$ to $t_j$, giving
\[ d^2 B_1^2 \int_{t_{j-1}}^{t_j} \| \partial_t \tilde{u}^j (t_j) \|^2 - t_{j-1}^2 \| \partial_t \tilde{u}^j (t_{j-1}) \|^2 \leq d^6 B_1^4 B_0^2 \varepsilon \int_{t_{j-1}}^{t_j} t^2 \| \nabla \tilde{u}^j \|^2 dt + d^2 B_1^2 \int_{t_{j-1}}^{t_j} t^2 \| \partial_t \tilde{f}^j \|^2 dt \]
\[ + d^2 B_1^2 \int_{t_{j-1}}^{t_j} t^2 \| \partial_t \tilde{u}^j \|^2 dt + 2d^2 B_1^2 \int_{t_{j-1}}^{t_j} t \| \partial_t \tilde{u}^j \|^2 dt. \]

(3.19)

Definition (2.10) and identity (2.4) imply
\[ \| \partial_t \tilde{u}^j (t_j) \| = \| \partial_t u (t_j) + b (t_{j-1}) \cdot \nabla u (t_{j-1}) \| = \| \partial_t \tilde{u}^{-1} (t_{j-1}) \|. \]

Therefore, summing up (3.19) we have
\[ \sum_{j=1}^{n} \varepsilon \int_{t_{j-1}}^{t_j} t^2 \| \nabla \partial_t \tilde{u}^j \|^2 dt - \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \| \nabla \tilde{u}^j \|^2 dt \]
\[ + d^2 B_1^2 T^2 \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \| \partial_t \tilde{f}^j \|^2 dt + d^2 B_1^2 (T + 2) \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} t \| \partial_t \tilde{u}^j \|^2 dt. \]

(3.20)

Recalling
\[ \| \partial_t \tilde{f}^j \| = \| \partial_t f + b \cdot \nabla f \| \leq \| \partial_t f \| + d^{1/2} B_0 \| \nabla f \|, \]

from (3.20) together with Lemmas 3.3 and 3.4 we conclude (3.17). \(\Box\)

If $u_0 \in H^2 \cap H^1_0$, we obtain in almost the same way, i.e., by combining energy methods applied in each interval $[t_{j-1}, t_j]$ with the transformation rules for the subcharacteristic coordinates, the following a priori estimates (see [2] for a detailed proof).

**LEMMA 3.6.** Suppose conditions (A1) and (A2) are satisfied and $u_0 \in H^2 \cap H^1_0$. For $n = 1, \ldots, N$ let $\tilde{u}^n$ be the solution of (3.2), (3.3) according to Theorem 3.1. Then there holds
\[ \sum_{j=1}^{n} \varepsilon \int_{t_{j-1}}^{t_j} \| \nabla \partial_t \tilde{u}^j (t) \|^2 dt \leq C_3 \left( \| u_0 \|_2^2 + \| f \|_C^2 (J, L^2) + \| f \|_C^2 (J, H^1) \right). \]

(3.21)

If additionally $u_0 \in D(A^{3/2})$ and $f(0) \in D(A^{1/2}) = H^1_0$ is satisfied, then there holds
\[ \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \| \partial_t^2 \tilde{u}^j (t) \|^2 dt \leq C_4 \left( \| u_0 \|_2^2 + \| \Delta u_0 \|_2^2 + \| f \|_C^2 (J, L^2) + \| f \|_C^2 (J, H^1) \right). \]

(3.22)

$C_3$ and $C_4$ depend on $d$, $T$, and $B_i$ being defined in (2.11) but not on $\varepsilon$. Additionally, $C_4$ depends on $A_\delta$ being defined in (2.16).

Inequality (3.22) is presented only in order to illustrate in section 4 the differences between our estimates and those given in the literature [9, 10, 28, 31]. To prove our main results (1.8)–(1.10), we do not use (3.22). If $u_0 \in H^2 \cap H^1_0$ and $f \in C(J, H^2 \cap H^1_0)$, then $\| \tilde{u}^n (t) \|_2$ remains uniformly bounded in $\varepsilon$. We prove this below for $f \equiv 0$ and refer to [2, p. 14] for $f \neq 0$. Using this result, (3.21) can be sharpened in the sense that for $f \equiv 0$ the bound on the right-hand side decreases quadratically if $\varepsilon \to 0$.\]
Lemma 3.7. Suppose condition (A1) is satisfied and \( u_0 \in H^2 \cap H_0^1 \) and \( f \equiv 0 \). For \( n = 1, \ldots, N \) let \( \tilde{u}^n \) be the solution of (3.2), (3.3) according to Theorem 3.1. Then there holds

\[
\sum_{j=1}^{n} \epsilon \int_{t_{j-1}}^{t_j} \| \nabla \partial_t \tilde{u}^j(t) \|^2 \, dt \leq C_5 \epsilon^2 \| u_0 \|^2_2.
\]

\( C_5 \) depends on \( d, T, B_i, \) and \( A_b, \) the latter being defined in (2.16).

Lemma 3.7 can be proven similarly to the lemmas above; see [2, p. 17] for details. Thus, it remains to establish an \( \epsilon \)-uniform bound for \( \| \tilde{u}^n(t) \|_2 \). First, we have the following lemma.

Lemma 3.8. Suppose condition (A1) is satisfied. For \( n = 1, \ldots, N \) and \( t_{n-1} \leq s \leq t \leq t_n \), \( A^n(t)U^n(t, s)A^n(s)^{-1} \) is a bounded mapping from \( L^2 \) into \( L^2 \) with

\[
\| A^n(t)U^n(t, s)A^n(s)^{-1} \| \leq (1 + 2 d^2 B_4 A_b k) e^{2 d^2 B_4 A_b k},
\]

where \( k = t_n - t_{n-1} \) and \( B_4 \) and \( A_b \) are defined in (2.11) and (2.16), respectively.

Proof. The case \( t = s \) is trivial. Suppose now \( t > s \). Let \( \{ e^{-tA^n(t)}; t \geq 0 \} \) denote the analytic semigroup generated by \(-A^n(t); \) see, e.g., [21]. Further, we set

\[
W(t, s) := A^n(t)U^n(t, s)A^n(s)^{-1}.
\]

Integrating this relation from \( s \) to \( t \), applying \( A^n(t) \) to the resulting equation and using that \( A^n(t) \) commutes on \( D(A^n(t)) \) with the semigroup, we get

\[
W(t, s)x = A^n(t)e^{-(t-s)}A^n(t)A^n(s)^{-1}x + \int_s^t A^n(t)e^{-(t-\tau)}A^n(t) (A^n(t) - A^n(\tau))A^n(\tau)^{-1}W(\tau, s)x \, d\tau.
\]

Since \( e^{-tA^n(t)} \) is contractive, we have \( \| e^{-tA^n(t)} \| \leq 1 \); see, e.g., [21]. The property

\[
\| A^n(t)e^{-(t-\tau)A^n(t)} \| \leq (t - \tau)^{-1}, \quad t > \tau,
\]

can easily be proven by the method of spectral resolution; cf. [12, Lemma 2.10]. From these two estimates for the semigroup and inequality (2.17) we deduce

\[
\| W(t, s)x \| \leq (1 + 2 d^2 B_4 A_b k) \| x \| + 2 d^2 B_4 A_b \int_s^t \| W(\tau, s)x \| \, d\tau.
\]

Finally, applying the Gronwall lemma to (3.23) yields

\[
\| W(t, s)x \| \leq (1 + 2 d^2 B_4 A_b k) e^{2 d^2 B_4 A_b (t-s)} \| x \|.
\]

Since \( t - s \leq k \) by assumption, we conclude the assertion. \( \square \)

Lemma 3.9. Suppose condition (A1) is satisfied and \( u_0 \in D(A) \) and \( f \equiv 0 \). For \( n = 1, \ldots, N \) let \( \tilde{u}^n \) be the solution of (3.2), (3.3) according to Theorem 3.1. Then there holds

\[
\| A^n \tilde{u}^n \|_{C(J_n; L^2)} \leq e^{4 d^2 B_4 A_b T} \| A u_0 \|,
\]
with $A_B$ and $A_B^b$ being defined in (2.11) and (2.16), respectively.

Proof. First, using Lemma 2.1, definition (2.5), and $Xj^{-1}(t_{j-1}) = x$, we note

$$A_j(t_{j-1})E_j^{-1}(t_{j-1}) = E_j(Au(t_{j-1})) = E_j(A_j^{-1}(t_{j-1})\tilde{u}^{-1}(t_{j-1})).$$

Therefore, arguing as in (3.8), it follows that $A^n(t)\tilde{u}^n(t)$ is expressible as

$$A^n(t)\tilde{u}^n(t) = A^n(t)U^n(t, t_n\{-1\}A^n(t_n-1)\tilde{u}^{-1}(t_{n-1}) \cdots A^n(t_1)U^n(t_0)\tilde{u}^{-1}(t_0)\tilde{u}.$$  

By Lemma 3.8 and identity (2.4) we get

$$\|A^n(t)\tilde{u}^n(t)\| \leq (1 + 2d^2A_B^b)k^n e^{2d^2A_B^b k} \|Au_0\|.$$

We recall, setting $x := 2d^2A_B^b T$, that $(1 + x)^N$ is monotonically increasing in $N$ and converges to $e^x$. Therefore, we conclude the assertion (3.24).  

If energy methods would be used to bound $\|A^n\tilde{u}^n\|_{L^2(J;L^2)}$ uniformly in $\varepsilon$, then (3.2) has to tested with $(A^n)^2\tilde{u}^n$, which would imply severer assumptions about $b$. Together, Lemma 3.9 and inequality (2.16) show that, if $u_0 \in H^2 \cap H^1_0$, then

$$\|\tilde{u}^n\|_{C(J;H^2)} \leq A_be^{d^2A_B^b T} \|u_0\|_2, \quad n = 1, \ldots, N.$$

4. Error estimates for the time discretization. In this section we derive a series of $\varepsilon$-uniform error estimates for the semidiscretization (1.4) in time of problem (1.1), (1.2). First, in the case of smooth initial values $u_0 \in H^2$ we have the following.

**Lemma 4.1.** Suppose conditions (A1) and (A2) are satisfied. Further assume $u_0 \in H^2 \cap H^1_0$. Let $u$ be the solution of (1.1), (1.2) according to Theorem 3.1, and let the sequence $(u^n)_n \subset H^1_0$ satisfy (1.4). Then for $n = 1, \ldots, N$ the error $e^n = u(t_n) - u^n$ satisfies

$$\|e^n\| + k^{1/2}\left(\sum_{j=1}^n \varepsilon \|\nabla e^j\|^2\right)^{1/2} \leq C_0 k(\|u_0\|_2 + \|f\|_{C(J;L^2)} + \|f\|_{C(J;H^1)}),$$

where $C_0 := 12 \max\{T \max\{1, d^{1/2}B_0\}, d \max\{B_5 C_3^{1/2}, dB_1B_0M_1^{1/2}\}\}$ with $M_1, C_3$, and $B_1$ being defined in Lemma 3.3, Lemma 3.6, and (2.11), respectively.

Proof. Combining (3.2) with (1.4) and recalling $X_j^j(t_j) = x$, we obtain

$$\langle e^j - e^{-j} \circ X_j^j(t_{j-1}), v \rangle + k \varepsilon \langle \nabla e^j, \nabla v \rangle = -\int_{t_{j-1}}^{t_j} (t - t_{j-1}) \partial_t^2 \tilde{u}^j dt, v \rangle$$

for $v \in H^2_0$. In contrast to other publications (see [9, 10, 28, 31]) we do not estimate the right-hand side of (4.1) by second order material derivates, i.e., by $k \int_{t_{j-1}}^{t_j} \|\partial_t^2 \tilde{u}^j\| dt \|v\|$, since this would ultimately imply, according to Lemma 3.6, the assumption $u_0 \in D(A^{3/2})$ in order to ensure $\varepsilon$-uniform convergence of first order for the time discretization. However, in computations we observed $\varepsilon$-uniform convergence of first order even for $H^2$-regular initial values. Hence, estimating the right-hand side of (4.1) by second order material derivates cannot be sharp. Instead, we put the relation (3.18) in the right-hand side of (4.1). Additionally setting $v = e^j$, this shows

$$\|e^j\|^2 + k \varepsilon \|\nabla e^j\|^2 \leq \frac{1}{2} \|e^j-1\|^2 + \frac{1}{2} \|e^j\|^2 + \int_{t_{j-1}}^{t_j} (t - t_{j-1}) |a^j_1(\partial_t \tilde{u}^j, e^j)| dt$$

$$+ \int_{t_{j-1}}^{t_j} (t - t_{j-1}) |\tilde{u}^j_1(\tilde{u}^j, e^j)| dt + \int_{t_{j-1}}^{t_j} (t - t_{j-1}) |(\tilde{u}^j_1, e^j)| dt.$$
By Cauchy–Young’s inequality we get

\[
\|e^j\|^2 + k \varepsilon \|\nabla e^j\|^2 \leq \frac{1}{2} \|e^{j-1}\|^2 + \frac{1}{2} \|e^j\|^2 + k^2 d^2 B_0^2 b_j + \frac{1}{4} k \varepsilon \|\nabla e^j\|^2 \\
+ k^2 d^2 B_0^2 c_j + \frac{1}{4} k \varepsilon \|\nabla e^j\|^2 + k^2 F\|e^j\|
\]

where \(b_j := \varepsilon \int_{t_{j-1}}^{t_j} \|\nabla \partial_t \tilde{u}^j\|^2 dt\), \(c_j := \varepsilon \int_{t_{j-1}}^{t_j} \|\nabla \tilde{u}^j\|^2 dt\), and \(F := \|f\|_{C^{1}(J;L^2)} + d^{1/2} B_0 \|f\|_{C(J;H^1)}\). We absorb \(ke\|\nabla e^j\|^2\) into the left-hand side of (4.2). Hence, using the Lagrangian framework results in a lower order norm of the solution, \(e \int_{t_{j-1}}^{t_j} \|\nabla \partial_t \tilde{u}^j\|^2 dt\) instead of \(\int_{t_{j-1}}^{t_j} \|\nabla^2 \tilde{u}^j\|^2 dt\), on the right-hand side of the error inequality (4.2). Summing up (4.2) and using Cauchy–Young’s inequality yields

\[
\|e^n\|^2 + 2 k \sum_{j=1}^{n} \varepsilon \|\nabla e^j\|^2 \leq h_0 + k \sum_{j=1}^{n-1} h_1 \|e^j\|,
\]

where \(h_0 := 4 k^2 (k^2 F^2 + d^2 \sum_{j=1}^{n} (B_0^2 b_j + B_0^2 c_j))\) and \(h_1 := 4 k F\).

Applying now the standard form of the discrete Gronwall lemma (see, e.g., [17, p. 369]), would imply an exponential dependence of the error constant on the final time \(T\). Generally, this strongly overestimates the real error. Instead, we use a nonlinear version of the discrete Gronwall lemma. We define the step functions

\[y(t) := \|e^j\| \quad \text{and} \quad z(t) := h_1\]

for \(t_{j-1} < t < t_j\) and \(j = 1, \ldots, n\). By virtue of (4.3) we thus have for \(t \in (t_{n-1}, t_n]\)

\[
\int_{0}^{t} z(s) y(s) ds = h_0 + \sum_{j=1}^{n-1} h_1 \|e^j\| + (t - t_{n-1}) h_1 \|e^n\| \geq (y(t))^2 - h_0.
\]

We further set

\[\hat{y}(t) = \int_{0}^{t} z(s) y(s) ds.
\]

Since \(\hat{y}\) is absolutely continuous, we find by (4.4) that

\[(d/dt)\hat{y}(t) = z(t) y(t) \leq z(t) (h_0 + \hat{y}(t))^{1/2}.
\]

Integrating this relation from 0 to \(t\) and noting \(\hat{y}(0) = 0\) gives

\[2(h_0 + \hat{y}(t))^{1/2} \leq 2 h_0^{1/2} + \int_{0}^{t} z(s) ds;\]

hence

\[h_0 + \hat{y}(t) \leq (h_0^{1/2} + \frac{1}{2} \int_{0}^{t} z(s) ds)^2.
\]

Now, for \(t \to t_{n-1}\) (\(t > t_{n-1}\)) we deduce from (4.5) that

\[
h_0 + \hat{y}(t) \leq (h_0^{1/2} + \frac{1}{2} \int_{0}^{t} z(s) ds)^2.
\]

Finally, combining (4.3) with relation (4.6) we get

\[
\|e^n\| + k^{1/2} \left(2 \sum_{j=1}^{n} \varepsilon \|\nabla e^j\|^2\right)^{1/2} \leq 2 h_0^{1/2} + h_1 (T - k),
\]
and thus
\[ \| e^n \| + k^{1/2} \left( 2 \sum_{j=1}^n \varepsilon \| \nabla e^j \|^2 \right)^{1/2} \]
\[ \leq 4kTF + 4kd \left( B_5 \left( \sum_{j=1}^n b_j \right) \right) \]
\[ + B_6 \left( \sum_{j=1}^n c_j \right)^{1/2}. \]
Together with Lemmas 3.3 and 3.5, this inequality proves Lemma 4.1. \( \square \)

In a similar way, using now Lemmas 3.7 and 3.9 instead of Lemmas 3.3 and 3.5, we obtain the following for a homogeneous function \( f \equiv 0 \) (see [2, p. 26] for a detailed proof).

**Lemma 4.2.** Suppose condition (A1) is satisfied. Assume \( u_0 \in H^2 \cap H^1_0 \) and \( f \equiv 0 \). Let \( u \) be the solution of (1.1), (1.2) according to Theorem 3.1 and let the sequence \( \{ u^n \}_{n=0}^N \subset H^1_0 \) satisfy (1.4). Then for \( n = 1, \ldots, N \) the error \( e^n = u(t_n) - u^n \) satisfies
\[ \| e^n \| + k^{1/2} \left( \sum_{j=1}^n \varepsilon \| \nabla e^j \|^2 \right)^{1/2} \]
\[ \leq C_7 \varepsilon k \| u_0 \|_2, \]
where \( C_7 \) depends on \( d, T, B_i \) and \( A_b \) with \( B_i \) and \( A_b \) being defined in (2.11) and (2.16), respectively.

Next, we consider the case of rough initial values \( u_0 \in L^2 \). We split the solution \( u \) of (1.1), (1.2) into two parts \( v \) and \( w \) in which the function \( v = v(t) \) satisfies (1.1), (1.2) with a homogeneous right-hand side \( f \equiv 0 \). The function \( w = w(t) \) is the solution of (1.1), (1.2) with homogeneous initial value \( u_0 = 0 \). In the same way, let \( \{ v^n \}_{n=0}^N \) satisfy the semidiscrete problem (1.4) with \( f \equiv 0 \) and \( \{ w^n \}_{n=0}^N \) satisfy (1.4) with \( u_0 = 0 \). Thus, \( u = v + w \) and \( u^n = v^n + w^n \), and the error \( \varepsilon^n := u(t_n) - u^n \) of the semidiscretization (1.4) is expressible as \( e^n = \xi^n + \eta^n \) with \( \xi^n := v(t_n) - v^n \) and \( \eta^n := w(t_n) - w^n \).

Further, we introduce the notation \( \tilde{v}^n := v \circ X^n \). Clearly, \( \tilde{v}^n \) denotes the transformation of the solution \( v \) into subcharacteristic coordinates, and \( v^n \) is the semidiscrete approximation in time of \( v \).

According to Lemma 4.1, for the error \( \eta^n = w(t_n) - w^n \) there holds
\[ \| \eta^n \|^2 + k \sum_{j=1}^n \varepsilon \| \nabla \eta^j \|^2 \leq C_8 k^2 \left( \| f \|^2_{C^1(J;L^2)} + \| f \|^2_{C(J;H^1)} \right) \]
with some constant \( C_8 = C_5(d, T, b) \) independent of \( \varepsilon \).

To estimate the error \( \xi^n = v(t_n) - v^n \), we first show the following.

**Lemma 4.3.** Suppose conditions (A1) and (A2) are satisfied. Let \( v \) be the solution of (1.1), (1.2) with \( f \equiv 0 \) according to Theorem 3.1, and let the sequence \( \{ v^n \}_{n=0}^N \subset H^1_0 \) satisfy the semidiscrete problem (1.4) with \( f \equiv 0 \). Then for \( n = 1, \ldots, N \) the error \( \xi^n = v(t_n) - v^n \) satisfies
\[ t_n^2 \| \xi^n \|^2 + k t_n^2 \sum_{j=1}^n \varepsilon \| \nabla \xi^j \|^2 \leq C_9 k^2 \| u_0 \|^2 + 2d^2 B_1^2 k e^{-n-1} \sum_{j=1}^{n-1} \| \xi^j \|^2_{L^2}, \]
where \( C_9 := 4d^2(B_1^2 C_2 + d^2 B_1^2 B_2^2 T^2 M_1) \) with \( B_i, M_1 \) and \( C_2 \) being defined in (2.11) and Lemmas 3.3 and 3.5, respectively.
Proof. Let \( \tilde{v}^j = v \circ X^j \). Then the differential equation (3.2) yields
\[
(4.10) \quad \langle \partial_t \tilde{v}^j(t_j), \phi \rangle + \varepsilon \langle \nabla \tilde{v}^j(t_j), \nabla \phi \rangle = 0
\]
for \( \phi \in H^1_0 \). Since \( X^j(t_j) = x \) and thus \( \tilde{v}^j(t_j) = v(t_j) \), from (4.10) and the weak form of the corresponding semidiscrete problem
\[
\langle v^j - v^{j-1} \circ X^j(t_{j-1}), \phi \rangle + k\varepsilon \langle \nabla v^j, \nabla \phi \rangle = 0, \quad \phi \in H^1_0,
\]
we obtain
\[
\langle \xi^j - \xi^{j-1} \circ X^j(t_{j-1}), \phi \rangle + k\varepsilon \langle \nabla \xi^j, \nabla \phi \rangle = -\langle \int_{t_{j-1}}^{t_j} (t - t_{j-1}) \partial_t^2 \tilde{v}^j dt, \phi \rangle.
\]
Putting \( \tilde{\xi}^j := t_j \xi^j \), we get
\[
\langle \tilde{\xi}^j, \phi \rangle + k\varepsilon \langle \nabla \tilde{\xi}^j, \nabla \phi \rangle = \langle \tilde{\xi}^{j-1} \circ X^j(t_{j-1}), \phi \rangle + k \langle \xi^{j-1} \circ X^j(t_{j-1}), \phi \rangle
\]
\[
- t_j \int_{t_{j-1}}^{t_j} (t - t_{j-1}) \langle \partial_t^2 \tilde{v}^j, \phi \rangle dt.
\]
Now, we substitute identity (3.18) with \( \tilde{f}^j \equiv 0 \) in the integral on the right-hand side of (4.11). Further, taking \( \phi = \tilde{\xi}^j \), we find
\[
||\tilde{\xi}^j||^2 + k\varepsilon ||\nabla \tilde{\xi}^j||^2 = ||\tilde{\xi}^{j-1} \circ X^j(t_{j-1}), \tilde{\xi}^j || + k \langle \xi^{j-1} \circ X^j(t_{j-1}), \tilde{\xi}^j ||
\]
\[
+ \int_{t_{j-1}}^{t_j} t_j(t - t_{j-1}) (\partial_t (\tilde{v}^j, \tilde{\xi}^j) + \tilde{a}_j (\tilde{v}^j, \tilde{\xi}^j)) dt.
\]
The uniqueness of the characteristic lines ensures that
\[
X(t_{j-1}; t_j, X(t_j; t_{j-1}, x)) = X(t_{j-1}; t_j, x) = x.
\]
Thus, by (2.3) there holds
\[
\langle \xi^{j-1} \circ X^j(t_{j-1}), \tilde{\xi}^j || = \langle \xi^{j-1} \circ X^j(t_{j-1}; t_j, \cdot), \tilde{\xi}^j || = \langle \xi^{j-1} , \tilde{\xi}^j \circ X(t_j; t_{j-1}, \cdot) \rangle.
\]
Further, for \( t_{j-1} \leq t \leq t_j \) we have \( t_j(t - t_{j-1}) \leq k t \). By some elementary inequalities we then deduce from (4.12) that
\[
||\tilde{\xi}^{j}||^2 + k\varepsilon ||\nabla \tilde{\xi}^{j}||^2 \leq \frac{1}{2} ||\tilde{\xi}^{j-1}||^2 + \frac{1}{2} ||\tilde{\xi}^{j}||^2 + 2d^2 \mathcal{B}_2 k^2 a_j + 2d^2 \mathcal{B}_2 k^2 T^2 b_j
\]
\[
+ \frac{1}{4} k \varepsilon ||\nabla \tilde{\xi}^{j}||^2 + d^2 \mathcal{B}_2 k \varepsilon^{-1} ||\xi^{j-1}||^2_{-1} + \frac{1}{4} k \varepsilon ||\nabla \tilde{\xi}^{j}||^2,
\]
where \( a_j := \varepsilon \int_{t_{j-1}}^{t_j} t^2 ||\nabla \partial_t \tilde{v}^j||^2 dt \) and \( b_j := \varepsilon \int_{t_{j-1}}^{t_j} ||\nabla \tilde{v}^j||^2 dt \). Summing up (4.13) and recalling \( \xi^0 = 0 \) gives
\[
||\tilde{\xi}^{j}||^2 + k \varepsilon \sum_{j=1}^{n} ||\nabla \tilde{\xi}^{j}||^2 \leq 4d^2 k^2 \sum_{j=1}^{n} (B_2^2 a_j + B_2^2 T^2 b_j) + 2d^2 \mathcal{B}_2 k \varepsilon^{-1} \sum_{j=1}^{n-1} ||\xi^{j}||^2_{-1}.
\]
Now employing Lemmas 3.3 and 3.5 with \( f \equiv 0 \) proves (4.9). \( \square \)

Consequently, it remains to estimate \( k \varepsilon^{-1} \sum_{j=1}^{n-1} ||\xi^{j}||^2_{-1} \).

**Lemma 4.4.** Suppose conditions \((A1)\) and \((A2)\) are satisfied. Let \( v \) be the solution of (1.1), (1.2) with \( f \equiv 0 \) according to Theorem 3.1, and let the sequence \( \{v^n\}_{n=0}^{N} \subset H^1_0 \)
satisfy the semidiscrete problem (1.4) with $f \equiv 0$. Then, for $n = 1, \ldots, N$ the error
\[
\xi^n = v(t_n) - v^n
\]
satisfies
\[
\|\xi^n\|_{-1} \leq C_{10} k^2 \|u_0\|, \quad k \leq 1, \quad n = 1, \ldots, N.
\]

where $C_{10} := \left(32d^2B_0^2A_0^2M_0^2(C_0 + T) + d^4B_0^2B_0^2M_1^2\right)e^{(128d^2B_0^2A_0^2M_0^2+1)T}$ with $B_0$, $A_0$, $M_0$, and $C_0$ being defined in (2.11), (2.16) and Lemmas 3.3 and 3.4, respectively.

Proof. From relations (1.1) and (1.4) we get
\[
\frac{d}{dt}\xi^j - \varepsilon \Delta \xi^j = -\frac{1}{k} \int_{t_{j-1}}^{t_j} (t - t_{j-1})\partial_t^2 \bar{v}^j dt, \quad j = 1, \ldots, n,
\]
where $d_t \xi^j := k^{-1}\left(\xi^j - \xi^j \circ X^j(t_{j-1})\right)$ is the discrete material derivative. We note that since $A = -\varepsilon \Delta$ with $D(A) := H^2 \cap H^1_0$ is m-sectorial, there holds $\{v^j\}_{j=1}^n \subset D(A)$. In terms of $T^j(t) := A^{-1}(t)^{-1}$, relation (4.16) can be rewritten as
\[
\frac{d}{dt}(T^j(t)\xi^j) + \xi^j = \Gamma_j := -T^j(t_j)\frac{1}{k} \int_{t_{j-1}}^{t_j} (t - t_{j-1})\partial_t^2 \bar{v}^j dt.
\]
Letting
\[
d_t(T^j(t_j)\xi^j) := -k^{-1}(T^j(t_j)\xi^j - T^j(t_{j-1})\xi^{j-1} \circ X^j(t_{j-1})),
\]
we immediately find
\[
\frac{d}{dt}(T^j(t_j)\xi^j) := -k^{-1}(T^j(t_j) - T^j(t_{j-1})),
\]
Taking the inner product with $T^j(t_j)\xi^j$ on both sides of (4.17) and using (4.18) yield
\[
\langle d_t(T^j(t_j)\xi^j), T^j(t_j)\xi^j \rangle + \langle \xi^j, T^j(t_j)\xi^j \rangle
\]
\[
= \langle \Gamma_j, T^j(t_j)\xi^j \rangle + \langle d_t(T^j(t_j))\xi^{j-1} \circ X^j(t_{j-1}), T^j(t_j)\xi^j \rangle.
\]
From Lemma 2.1 we get
\[
(A\psi) \circ X^j(t) = A^j(t)(\psi \circ X^j(t)), \quad \psi \in D(A),
\]
which directly implies that
\[
(T\psi) \circ X^j(t) = T^j(t)(\psi \circ X^j(t)), \quad \psi \in L^2,
\]
where $T := A^{-1}$. By (4.21) and elementary calculations we obtain
\[
2k \langle d_t(T^j(t_j)\xi^j), T^j(t_j)\xi^j \rangle = \|T^j(t_j)\xi^j\|^2 - \|T^{j-1}(t_{j-1})\xi^{j-1}\|^2 + k^2\|d_t(T^j(t_j)\xi^j)\|^2.
\]
Therefore, (4.19) can be written as
\[
\|T\xi^j\|^2 - \|T^{j-1}\|^2 + k^2\|d_t(T^j(t_j)\xi^j)\|^2 + 2k\|T^{j-1}\xi^{j-1}\|^2.
\]
\[
= 2k \langle \Gamma_j, T\xi^j \rangle + 2k \langle d_t(T^j(t_j))\xi^{j-1} \circ X^j(t_{j-1}), T\xi^j \rangle.
\]
We now estimate the right-hand side of (4.22). Since $f \equiv 0$ by assumption, putting $\phi = T^2\xi^j$ in (3.18) gives
\[
k \langle \Gamma_j, T\xi^j \rangle \leq \int_{t_{j-1}}^{t_j} (t - t_{j-1}) \langle \partial_t^2 \bar{v}^j, T^2\xi^j \rangle dt
\]
\[
\leq \int_{t_{j-1}}^{t_j} (t - t_{j-1}) |a_t^j(\partial_t\bar{v}^j, T^2\xi^j)| dt + \int_{t_{j-1}}^{t_j} (t - t_{j-1}) |a_t^j(\bar{v}^j, T^2\xi^j)| dt.
\]
Since $\partial_t \tilde{v}^j(t) + A^j(t)\tilde{v}^j(t) = 0$, it follows that
\[
|a^j(t)(\partial_t \tilde{v}^j, T^2\xi^j)| = |(\partial_t \tilde{v}^j, A^j(t)T^j(t)T\xi^j)|
\leq |(\partial_t \tilde{v}^j, (A^j(t) - A^i(t_j))T^j(t_j)T\xi^j)| + |(\partial_t \tilde{v}^j, A^i(t_j)T^j(t_j)T\xi^j)|
= |(\partial_t \tilde{v}^j, (A^j(t) - A^i(t_j))T^j(t_j)T\xi^j)| + |(D^j\nabla \tilde{v}^j, \nabla T\xi^j)|
\]
and, moreover, by (2.17) that
\[
|a^j(t)(\partial_t \tilde{v}^j, T^2\xi^j)| \leq 2d^2B_4A_b(t_j - t)|\partial_t \tilde{v}^j||T\xi^j| + dB_5\varepsilon^{1/2}|\nabla \tilde{v}^j||\varepsilon^{1/2}||\nabla T\xi^j||
= 2d^2B_4A_b(t_j - t)|\partial_t \tilde{v}^j||T\xi^j| + dB_5\varepsilon^{1/2}|\nabla \tilde{v}^j||T^{1/2}\xi^j|.
\]
Thus, we have
\[
(t - t_{j-1})|a^j(t)(\partial_t \tilde{v}^j, T^2\xi^j)|
\leq 16d^4B_4^2A_b^2k^3T|\partial_t \tilde{v}^j|^2 + \frac{1}{16}||T\xi^j||^2 + \frac{1}{2}d^2B_5^2k^2\varepsilon|\nabla \tilde{v}^j|^2 + \frac{1}{2}||T^{1/2}\xi^j||^2.
\]
Further, integrating by parts and using (2.16) we get
\[
|\dot{a}^j(t)(\tilde{v}^j, T^2\xi^j)| \leq ||\tilde{v}^j||||\nabla \cdot (\tilde{D}\nabla T^2\xi^j)|| \leq 2d^2\varepsilon B_4||\tilde{v}^j||T^2\xi^j||_2 \leq 2d^2B_4A_b||\tilde{v}^j||T\xi^j||,
\]
such that
\[
(t - t_{j-1})|\dot{a}^j(t)(\tilde{v}^j, T^2\xi^j)| \leq 16d^4B_4^2A_b^2k^2||\tilde{v}^j||^2 + \frac{1}{16}||T\xi^j||^2.
\]
For the second term on the right-hand side of (4.22) we obtain by the mean value theorem (see, e.g., [24, p. 3]) that
\[
\langle (dtT^j(t_j))\xi^{j-1} \circ X^j(t_{j-1}), T\xi^j \rangle \leq ||T^{j'}(t_*)\xi^{j-1} \circ X^j(t_{j-1})|| ||T\xi^j||
\]
with some $t_* \in (t_{j-1}, t_j)$. Since $T^{j'}(t) = -A^j(t)^{-1}A^{j'}(t)A^j(t)^{-1}$ (see, e.g., [13, p. 130]), we find by (2.16) that
\[
||T^{j'}(t_*)\xi^{j-1} \circ X^j(t_{j-1})|| = \sup_{0 \neq \phi \in L^2} \left\{ \frac{\langle T^{j'}(t_*)\xi^{j-1} \circ X^j(t_{j-1}), \phi \rangle}{||\phi||} \right\}
= \sup_{0 \neq \phi \in L^2} \left\{ \frac{\langle T^j(t_{j-1})\xi^{j-1} \circ X^j(t_{j-1}), -A^j(t_{j-1})A^{j'}(t_*)^{-1}A^{j'}(t_*)^{-1}\phi \rangle}{||\phi||} \right\}
\leq 2d^2B_3A_b \sup_{0 \neq \phi \in L^2} \left\{ \frac{\langle T^j(t_{j-1})\xi^{j-1} \circ X^j(t_{j-1}), ||A^{j'}(t_*)A^j(t_*)^{-1}\phi|| \rangle}{||\phi||} \right\}
\leq 4d^4B_3A_b^2||T^j(t_{j-1})\xi^{j-1} \circ X^j(t_{j-1})||.
\]
Further recalling (2.4) and (4.21), we obtain
\[
k||((dtT^j(t_j))\xi^{j-1} \circ X^j(t_{j-1}), T\xi^j)|| \leq 32d^8B_3^2B_4^2A_b^2k||T\xi^{j-1}||^2 + \frac{1}{8}k||T\xi^j||^2.
\]
Now we sum up (4.22) and use the previous estimates with an obvious modification for $j = n$. Recalling $\xi^0 = 0$, we get
\[
||T\xi^n||^2 + k \sum_{j=1}^n ||T^{1/2}\xi^j||^2 \leq \frac{1}{2}||T\xi^n||^2 + \left(64d^8B_4^2B_4^2A_b^2M_2 + \frac{1}{2}\right) \sum_{j=1}^{n-1} k||T\xi^j||^2 + 32d^4B_4^2A_b^2M_2k^2 \sum_{j=1}^n (a_j + c_j) + d^2B_5^2k^2 \sum_{j=1}^n b_j,
\]
where $a_j := \int_{t_{j-1}}^{t_j} t \|\partial_t \hat{v}^j\|^2 \, dt$, $b_j := \varepsilon \int_{t_{j-1}}^{t_j} \|\nabla \hat{v}^j\|^2 \, dt$, and $c_j := \int_{t_{j-1}}^{t_j} \|\hat{v}^j\|^2 \, dt$. By the discrete version of the Gronwall lemma (see [17, p. 369]) it follows that

$$k \sum_{j=1}^{n} \|T^{1/2} \xi^j\|^2 \leq k^2 \left( 32 d^4 B_4^2 A_5^2 M_2^2 \sum_{j=1}^{n} (a_j + c_j) + d^2 B_2^2 \sum_{j=1}^{n} b_j \right) e^{(128 d^6 B_4^2 B_4^2 A_5^2 M_2^2 + 1)T}.$$  

We note that

$$\varepsilon^{-1} \|\xi^j\|_1^2 \leq \|T^{1/2} \xi^j\|^2,$$

which is a direct consequence of

$$\langle \xi^j, \phi \rangle = \langle T^{1/2} \xi^j, T^{-1/2} \phi \rangle = \langle T^{1/2} \xi^j, A^{1/2} \phi \rangle \leq \varepsilon^{1/2} \|T^{1/2} \xi^j\| \|\nabla \phi\|, \quad \phi \in H^1_0.$$  

Therefore, we find

$$k \varepsilon^{-1} \sum_{j=1}^{n} \|\xi^j\|_1^2 \leq k^2 \left( 32 d^4 B_4^2 A_5^2 M_2^2 \sum_{j=1}^{n} (a_j + c_j) + d^2 B_2^2 \sum_{j=1}^{n} b_j \right) e^{(128 d^6 B_4^2 B_4^2 A_5^2 M_2^2 + 1)T}.$$  

Now applying Lemmas 3.2 to 3.4 gives the assertion (4.15). □

Together, Lemmas 4.3 and 4.4 show the following.

**Lemma 4.5.** Suppose conditions (A1) and (A2) are satisfied. Let $v$ be the solution of (1.1), (1.2) with $f \equiv 0$ according to Theorem 3.1, and let the sequence $\{v^n\}_{n=0}^{N} \subset H^1_0$ satisfy the semidiscrete problem (1.4) with $f \equiv 0$. Then, for $n = 1, \ldots, N$ the error $\xi^n = v(t_n) - v^n$ satisfies

$$t_n^2 \|\xi^n\|^2 + k t_n^2 \sum_{j=1}^{n} \varepsilon \|\nabla \xi^j\|^2 \leq C_{11} k^2 \|u_0\|^2,$$

where $C_{11} := C_9 + 2 d^2 B_4^2 C_{10}$ with $C_9$ and $C_{10}$ being defined in Lemmas 4.3 and 4.4.

Finally, combining estimate (4.8) with Lemma 4.5, we have proven the following.

**Lemma 4.6.** Suppose conditions (A1) and (A2) are satisfied. Let $u$ be the solution of (1.1), (1.2) according to Theorem 3.1, and let the sequence $\{u^n\}_{n=0}^{N} \subset H^1_0$ satisfy the semidiscrete problem (1.4). Then, for $n = 1, \ldots, N$ the error $e^n := u(t_n) - u^n$ of the semidiscretization (1.4) in time satisfies

$$(4.24) \quad t n \|e^n\| + t_n k^{1/2} \left( \sum_{j=1}^{n} \varepsilon \|\nabla e^j\|^2 \right)^{1/2} \leq C_{12} k \left( \|u_0\| + \|f\|_{C^1(J;L^2)} + \|f\|_{C(J;H^1)} \right)$$

with $C_{12} := 2 \max\{C_{12}^{1/2}, C_{11}^{1/2}\}$ and $C_9, C_{11}$ being defined in (4.8) and Lemma 4.5.

5. Finite element approximation and technical preliminaries. In this section we first introduce basic facts about the finite element discretization. Next, we provide and recall some fundamental results of complex interpolation of Banach spaces. In section 6 we then prove two scales of $\varepsilon$-uniform a priori estimates for the spatial discretization (1.5) of the semidiscrete problem (1.4). We will observe that the $\varepsilon$-uniform error estimate of full order

$$\|u^n - u^h\| \leq c(h^2 + \min\{h, h^2/k\}),$$
which we prove below, requires severe regularity assumptions about the data \( u_0 \) and \( f \). Therefore, by using the method of complex interpolation of Banach spaces, we derive suboptimal \( \varepsilon \)-uniform error estimates under less restrictive conditions about the data.

We start with assumptions about the finite element discretization. Let \( \Pi_h = \{ K \} \) be a finite decomposition of mesh size \( h \) of the polyhedral domain \( \bar{\Omega} \) into closed subsets \( K \), triangles or quadrilaterals in two dimensions, tetrahedrons or hexahedrons in three dimensions, and, for simplicity, \( d \)-simplices if \( d \geq 4 \). The decompositions \( \Pi_h \) are assumed to be "face to face" and to satisfy a "uniform size" condition:

(B1). Any two elements of \( \Pi_h \) meet only on entire common faces or sides or in vertices. Each element of \( \Pi_h \) contains a \( d \)-ball of radius \( \kappa_1 h \) and is contained in a \( d \)-ball of radius \( \kappa_2 h \).

Restricting to polyhedral domains simplifies the arguments below in that the decompositions \( \Pi_h \), consisting of "straight" elements, exactly cover \( \bar{\Omega} \). However, assumption (A1) is valid for any smoothly bounded domain, and to preserve our results for such domains, it suffices to approximate the boundary to order \( O(h^2) \) by a (not necessarily convex) polygon or polyhedron. The theorems may be extended to smoothly bounded domains using standard techniques for dealing with such boundary approximation. One may further adapt the mesh to the boundary by using isoparametric finite elements. For an analysis of this procedure we refer to the literature; see, e.g., [7].

The finite element space to be used for approximating the "solution space" \( H^1_0 \) is

\[ H_h = \{ \phi_h \in C(\bar{\Omega}) \mid \phi_h|_K \in P(K) \text{ for all } K \in \Pi_h \} \cap H^1_0. \]

Here, \( P(K) \) is a space of polynomials, the maximum degree of which is bounded uniformly with respect to \( K \in \Pi_h \) and \( h \). If \( \Omega \) is decomposed into \( d \)-simplices, \( P(K) \) may consist of all polynomials of degree less or equal to one. We further assume the following condition:

(B2). For each \( v \in H^2 \cap H^1_0 \) there exists an approximation \( i_h v \in H_h \), such that

\[ \| v - i_h v \| + h \| \nabla (v - i_h v) \| \leq \kappa_3 h^2 \| v \|_2. \]

These are standard conditions satisfied by all the usual spaces of conforming finite elements (see, e.g., [7]). An extension of the error analysis for the spatial discretization to nonconforming finite elements is straightforward (cf. [6]).

The condition (B1) ensures that piecewise polynomial functions satisfy the usual "inverse relations," in particular (see [7])

\[ \| \nabla \phi_h \| \leq \kappa_4 h^{-1} \| \phi_h \|, \quad \phi_h \in H_h. \]

Throughout, \( P_h : L^2 \mapsto L^2 \) denotes the standard \( L^2 \)-projection onto \( H_h \). Then, we easily conclude from assumption (B2) that

\[ \| v - P_h v \| + h \| \nabla (v - P_h v) \| \leq \kappa_5 h^2 \| v \|_2, \quad v \in H^2 \cap H^1_0, \]

where \( \kappa_5 := \kappa_3 (2 + \kappa_4) \). In our Lagrangian framework, the elliptic or Ritz projection \( R^1_h(t) : H^1_0 \mapsto H_h \), \( t \in [t_{j-1}, t_j] \) is defined through the relation

\[ \langle D^i(t) \nabla (v - R^1_h(t)v), \nabla \phi_h \rangle = 0, \quad v \in H^1_0, \]

for all \( \phi_h \in H_h \). Recalling \( D^i(t) = (\nabla X^i(t))^{-1} \varepsilon I (\nabla X^i(t))^{-T} \), we observe that the Ritz projection \( R^1_h(t) \) incorporates the particle trajectories and thus the characteristics of the flow. Since \( D^i(t_j) = \varepsilon I \), the operator \( R_h := R^1_h(t_j) \) satisfies

\[ \langle \varepsilon \nabla (v - R_h v), \nabla \phi_h \rangle = 0, \quad v \in H^1_0, \]
for all \( \phi_h \in H_h \). From assumption (B2) we easily deduce by standard techniques that
\[
\|v - R_h v\| + h \| \nabla (v - R_h v)\| \leq \kappa_6 h^2 \| v \|_2, \quad v \in H^2 \cap H_0^1,
\]
where \( \kappa_6 := \kappa_3 (1 + A(1,1) \kappa_3) \) with \( A(\cdot, \cdot) \) being defined in assumption (A1).

Next, we recall a few basic notations in the complex interpolation theory of Banach spaces. Given an interpolation couple \( \{X_0, X_1\} \) of complex Banach spaces, \( F(X_0, X_1) \) denotes the space of all functions \( g(z) \) defined to be continuous from the closed strip \( \{0 \leq \text{Re} \, z \leq 1\} \) of the complex plane into \( X_0 + X_1 \), analytic in the interior \( \{0 < \text{Re} \, z < 1\} \), and such that the maps \( t \mapsto g(j + It) \), \( j = 0, 1 \), are bounded and continuous from \( \mathbb{R} \) to \( X_j \). Here, \( I \) is the imaginary unit and \( X_0 + X_1 \) is the Banach space \( \{y = x_0 + x_1 \mid x_j \in X_j, \, j = 0, 1\} \) with norm
\[
\|y\|_{X_0 + X_1} := \inf \{\|x_0\|_{X_0} + \|x_1\|_{X_1} \mid y = x_0 + x_1\}.
\]
By the three-lines theorem, \( F = F(X_0, X_1) \) is a Banach space in the norm
\[
|g|_F := \max \{\sup_{t \in \mathbb{R}} \|g(\text{It})\|_{X_0}, \sup_{t \in \mathbb{R}} \|g(1 + \text{It})\|_{X_1}\}.
\]
By \( [X_0, X_1]_\theta \), with \( 0 \leq \theta \leq 1 \), we denote the complex interpolation space between \( X_0 \) and \( X_1 \) with norm
\[
|v|_\theta := \inf \{|g|_F \mid g \in F(X_0, X_1), \, g(\theta) = v\}.
\]
For basic facts in complex interpolation theory we refer to [23] or [33]. In what follows the complexifications of various function spaces will be written with the same notation as the original real ones.

In section 6 we employ the following interpolation results.

**Lemma 5.1.** Suppose condition (A1) is satisfied. If \( 0 < \theta < 1/4 \), then (with equivalent norms) \( H^{2\theta} = [L^2, H^{2\theta} \cap H_0^1]_{\theta} \). If \( 1/4 < \theta \leq 1/2 \), then \( H_{2\theta}^{2\theta} = [L^2, H^{2\theta} \cap H_0^1]_{\theta} \). If \( 1/2 < \theta < 1 \), then \( H^{2\theta} \cap H_0^1 = [L^2, H^{2\theta} \cap H_0^1]_{\theta} \), where \( H^{2\theta} \cap H_0^1 \) is equipped with the norm \( \| \cdot \|_{2\theta} \).

Only in order to simplify the notation do we exclude throughout the case \( \theta = 1/4 \). We recall that \( v \in [L^2, H_0^1]_{1/2} \) imposes a boundary condition on \( v \), whereas \( v \in H_0^{1/2} \) does not, and that the norm of \( [L^2, H_0^1]_{1/2} \) is stricter than the Slobodeckij norm of \( H_0^{1/2} \); cf. [25, p. 66]. Hence, to preserve the error estimates below for \( \theta = 1/4 \), it suffices to replace the Slobodeckij norm by the norm of \( [L^2, H_0^1]_{1/2} \) or an equivalent norm.

The relation
\[
H_{2\theta}^{2\theta} = [L^2, H_0^{2\theta}]_{2\theta}, \quad 0 < \theta < 1/2, \, \theta \neq 1/4,
\]
is proven in [25, p. 64] in a more general context. In [25], however, \( \partial \Omega \) is supposed to be a \((d - 1)\)-dimensional infinitely differentiable variety. But this assumption is not essential. The property (5.6) can be shown for bounded domains with Lipschitz boundaries in almost exactly the same way as in [25]. We further recall that \( H_0^{2\theta} = H^{2\theta} \) for \( 0 < \theta < 1/4 \); see [14, p. 31]. Then, by the interpolation theory for self-adjoint operators (see [33, p. 141]) and the reiteration theorem (see [23, p. 232]), we easily deduce the first two relations of Lemma 5.1; see [4] or [1, Section 3.1] for details. Additionally employing the interpolation theorem for subspaces (see [33, p. 118]) yields the last identity; cf. [4] or [1, Section 3.1].

Combining the previous lemma with the interpolation between two spaces of continuous functions with Hilbert range (see [25, p. 95]), we obtain the following.
Lemma 5.2. Suppose condition (A1) is satisfied. If \( 0 < \theta < \frac{1}{4} \), then (with equivalent norms) \( C(J; H^{2\theta}) = [C(J; L^2), C(J; H^{2\theta} \cap H^0)]_\theta \). If \( \frac{1}{4} \leq \theta < \frac{1}{2} \), then \( C(J; H^{2\theta}) = [C(J; L^2), C(J; H^{2\theta} \cap H^0)]_\theta \). If \( \frac{1}{2} < \theta < 1 \), then \( C(J; H^{2\theta} \cap H^0) = [C(J; L^2), C(J; H^{2\theta} \cap H^0)]_\theta \).

Finally, we provide some \( \varepsilon \)-uniform a priori estimates for the solution \( \{u^n\}_{n=1}^N \) and \( \{u^n_h\}_{n=1}^N \) of the semidiscrete and fully discrete problem, respectively.

Lemma 5.3. Suppose condition (A1) is satisfied. Assume further \( u_0 \in L^2 \) and \( f \in C(J; L^2) \). Then, for the unique solution \( \{u^n\}_{n=1}^N \subset H^2 \cap H^0 \) of the semidiscrete problem (1.4) we have

\[
\|u^n\| \leq \|u_0\| + T \|f\|_{C(J; L^2)}
\]

for \( n = 1, \ldots, N \). If \( u_0 \in H^2 \cap H^1 \) and \( f \in C(J; H^2 \cap H^0) \), there holds

\[
\|u^n\|_2 \leq A_0 e^{2^\theta B_4 A_0 T} (\|u_0\|_2 + T \|f\|_{C(J; H^2)})
\]

with \( B_4 \) and \( A_0 \) being defined in (2.11) and (2.16), respectively.

Proof. Since the operator \( A = -\varepsilon \Delta \) is \( m \)-sectorial, the semidiscrete problem (1.4) has a unique solution \( \{u^n\}_{n=1}^N \subset H^2 \cap H^0 \). Moreover, we find

\[
u^n = (1 + kA)^{-1} E^n u^{n-1} + k(1 + kA)^{-1} f(t_n)
\]

\[
= (1 + kA)^{-1} E^n (1 + kA)^{-1} E^{n-2} (1 + kA)^{-1} E^n (1 + kA)^{-1} f(t_{n-1}) + k(1 + kA)^{-1} f(t_n)
\]

(5.9)

(5.8)

Recalling \( \|1 + kA\| \leq 1 \), (2.4), and (2.5), we conclude from (5.9) the assertion (5.7).

Now suppose \( u_0 \in H^2 \cap H^1 \) and \( f \in C(J; H^2 \cap H^0) \). Applying \( A \) on both sides of (5.9) yields

\[
Au^n = A(1 + kA)^{-1} E^n (1 + kA)^{-1} E^1 u_0 + kA(1 + kA)^{-1} f(t_n)
\]

\[
+ k \sum_{j=1}^{n-1} A(1 + kA)^{-1} E^n (1 + kA)^{-1} E^{j+1} (1 + kA)^{-1} f(t_j).
\]

The operators \( A \) and \( (1 + kA)^{-1} \) are commutable on the domain of \( A \). Further, we observe that, in virtue of the transformation rule (4.20),

\[
(1 + kA)^{-1} A E^j = (1 + kA)^{-1} A A^j (t_{j-1})^{-1} A^j (t_{j-1})^{-1} E^j = (1 + kA)^{-1} A A^j (t_{j-1})^{-1} E^j A.
\]

Next, we express \( A A^j (t_{j-1})^{-1} \) as \( A A^j (t_{j-1})^{-1} = (A^j (t_{j}) - A^j (t_{j-1})) A^j (t_{j-1})^{-1} + I \), where \( I \) denotes the identity operator. Therefore, we find

\[
Au^n = (1 + kA)^{-1} ((A^n (t_n) - A^n (t_{n-1})) A^n (t_{n-1})^{-1} + I) E^n
\]

\[
\cdots (1 + kA)^{-1} ((A^1 (t_1) - A^1 (t_0)) A^1 (t_0)^{-1} + I) E^1 A u_0 + k(1 + kA)^{-1} A f(t_n)
\]

\[
+ k \sum_{j=1}^{n-1} (1 + kA)^{-1} ((A^n (t_n) - A^n (t_{n-1})) A^n (t_{n-1})^{-1} + I) E^n
\]

\[
\cdots (1 + kA)^{-1} ((A^{j+1} (t_{j+1}) - A^{j+1} (t_{j})) A^{j+1} (t_j)^{-1} + I) E^{j+1} (1 + kA)^{-1} A f(t_j).
\]
Since (cf. (2.17))
\[
\|(A^j(t_j) - A^j(t_{j-1}))A^j(t_{j-1})^{-1} + I\| \leq 1 + 2d^2B_4A_bk
\]
and \( T = Nk \), the right-hand side of the previous formula is estimated from above by
\[
\|A\eta^n\| \leq \left( 1 + \frac{2d^2B_4A_bT}{N} \right)^N (\|A\eta_0\| + t_n\|Af\|_{C(J;L^2)}) \\
\leq e^{2d^2B_4A_bT} (\|A\eta_0\| + t_n\|Af\|_{C(J;L^2)}).
\]

Thus, we conclude by (2.16) that
\[
\|\eta^n\|_2 \leq A_b^{-1}\|A\eta^n\| \leq A_b e^{2d^2B_4A_bT} (\|\eta_0\| + t_n\|f\|_{C(J;L^2)}).
\]

This completes the proof. \( \square \)

By a similar argument we obtain the following for the fully discrete problem (1.5).

**Lemma 5.4.** Suppose conditions (B1) and (B2) are satisfied. Assume further \( \eta_0 \in L^2 \), \( f \in C(J;L^2) \), and \( \eta^n_0 = P_h\eta_0 \). Then for the unique solution \( \{\eta^n\}_{n=1}^N \subset H_h \) of problem (1.5) there holds
\[
\|\eta^n\| \leq (1 + 2d^2B_4A_bT) (\|\eta_0\| + t_n\|f\|_{C(J;L^2)}).
\]

6. Error estimates for the spatial discretization. We are now in a position to estimate the error of the finite element approximation (1.5) of the semidiscrete problem (1.4). First, we show the \( k \)-uniform alternative contained in (1.8)-(1.10).

**Lemma 6.1.** Suppose conditions (A1), (B1), and (B2) are satisfied. Assume \( \eta_0 \in H^2 \cap H_0^1 \), \( f \in C(J;H^2 \cap H_0^1) \), and \( \eta^n_0 = P_h\eta_0 \). Let \( \{\eta^n\}_{n=1}^N \subset H^2 \cap H_0^1 \) be the solution of (1.4) and \( \{\eta^n_h\}_{n=1}^N \subset H_h \) of (1.5). Then for \( n = 1, \ldots, N \) there holds
\[
\|\eta^n - \eta^n_h\| \leq C_{13}(h^2 + h) (\|\eta_0\| + t_n\|f\|_{C(J;H^2)}),
\]
where \( C_{13} := \max\{1, 2e^{1/2T^{1/2}} + 2d^{1/2B_0T}\} \kappa_5A_bM_2 e^{2d^2B_4A_bT} \) with \( B_i, A_b, \kappa_5, \) and \( M_2 \) being defined in (2.11), (2.16), (5.2), and Lemma 3.3, respectively.

**Proof.** We split the error \( \eta^n - \eta^n_h \) into two parts:
\[
\eta^n - \eta^n_h = (\eta^n - P_h\eta^n) + (P_h\eta^n - \eta^n_h).
\]

According to (5.2), for the first term we have
\[
\|\eta^n - P_h\eta^n\| + h \|\nabla(\eta^n - P_h\eta^n)\| \leq \kappa_5h^2 \|\eta^n\|_2.
\]

To estimate \( \|P_h\eta^n - \eta^n_h\| \), we take the inner product with \( \phi_h \in H_h \) on both sides of (1.4), and combine the resulting equation with (1.5) to find
\[
\langle P_h w^j - w^j_h, \phi_h \rangle + k \varepsilon \langle \nabla(P_h w^j - w^j_h), \nabla \phi_h \rangle = k \varepsilon \langle \nabla(P_h w^j - w^j), \nabla \phi_h \rangle + \langle (w^{j-1} - w^{j-1}_h) \circ X^j(t_{j-1}), \phi_h \rangle \\
= k \varepsilon \langle \nabla(P_h w^j - w^j), \nabla \phi_h \rangle + \langle (P_h w^{j-1} - w^{j-1}_h) \circ X^j(t_{j-1}), \phi_h \rangle \\
+ \langle (P_h w^{j-1} - w^{j-1}) - (P_h w^{j-1} - w^{j-1}) \circ X^j(t_{j-1}), \phi_h \rangle.
\]

Putting \( \phi_h = P_h w^j - w^j_h \), we get by (2.4) and some elementary inequalities
\[
\|P_h w^j - w^j_h\|^2 \leq k \varepsilon \|\nabla(P_h w^j - w^j_h)\|^2 \leq k \varepsilon \|\nabla(w^j - P_h w^j)\|^2 + \|P_h w^{j-1} - w^{j-1}_h\|^2 \\
+ 2 \| (P_h w^{j-1} - w^{j-1}) - (P_h w^{j-1} - w^{j-1}) \circ X^j(t_{j-1})\| \|P_h w^j - w^j_h\|.
\]
Recalling $X^j(t_j) = x$, we observe that for any space function $\phi \in H^1$ there holds

$$\|\phi - \phi \circ X^j(t_{j-1})\| \leq \int_{t_{j-1}}^{t_j} \|\partial_t (\phi \circ X^j(t))\| \, dt \leq \int_{t_{j-1}}^{t_j} \| (b(t) \cdot \nabla \phi) \circ X^j(t)\| \, dt.$$ 

By (2.4) and (6.2) we thus have

$$\| (P_h u^{j-1} - u^{j-1}) - (P_h u^{j-1} - u^{j-1}) \circ X^j(t_{j-1})\| \leq k \frac{d}{2} B_0 \kappa_5 h \| u^{j-1} \|_2.$$

Now, we combine the previous estimates and sum up the resulting inequality. Since $u^0_h = P_h u_0$ and $u^0 = u_0$ by assumption, we obtain

$$\| P_h u^n - u^n_h \|^2 \leq k \kappa^2 h^2 \sum_{j=1}^{n} \| u^j \|^2_2 + k \sum_{j=1}^{n} 2 d^1/2 B_0 \kappa_5 h \| u^{j-1} \|_2 \| P_h u^j - u^j_h \|.$$

By Cauchy–Young’s inequality and the a priori estimate (5.8) it follows that

$$\| P_h u^n - u^n_h \|^2 \leq g_0 h^2 + k \sum_{j=1}^{n-1} g_1 \| P_h u^j - u^j_h \|,$$

where $g_0 := 2 \kappa^2 (\varepsilon + 2 d B_0^2 h^2) A_0^2 \varepsilon d^2 B_1 A_k T (\| u^0 \|_2 + T \| f \|_{C(J; H^2)})^2$ and $g_1 := (4 h d^1/2 B_0 \kappa_5 A_0 c^2 d^2 B_1 A_k T (\| u^0 \|_2 + T \| f \|_{C(J; H^2)})).$ Next, we define the step functions

$$y(t) := \| P_h u^j - u^j_h \| \quad \text{and} \quad z(t) := g_1$$

for $t_{j-1} < t \leq t_j$ and $j = 1, \ldots, n$. By virtue of (6.3) we thus have for $t \in (t_{n-1}, t_n]$

$$\int_0^t z(s) y(s) \, ds = k \sum_{j=1}^{n-1} g_1 \| P_h u^j - u^j_h \| + (t - t_{n-1}) g_1 \| P_h u^n - u^n_h \| \geq (y(t))^2 - g_0 h^2.$$

We set $\tilde{y}(t) = \int_0^t z(s) y(s) \, ds$ and argue as in the proof of Lemma 4.1 to find

$$g_0 h^2 + \tilde{y}(t) \leq (g_0^{1/2} h + \frac{1}{2} \int_0^t z(s) \, ds)^2.$$

Now, for $t \to t_{n-1}$ ($t > t_{n-1}$) we deduce from (6.4) that

$$g_0 h^2 + k \sum_{j=1}^{n-1} g_1 \| P_h u^j - u^j_h \| \leq \left( g_0^{1/2} h + \frac{1}{2} k \sum_{j=1}^{n-1} g_1 \right)^2 \leq (g_0^{1/2} h + \frac{1}{2} g_1 (T - k))^2.$$

Therefore, we conclude

$$\| P_h u^n - u^n_h \| \leq g_2 h (\| u^0 \|_2 + \| f \|_{C(J; H^2)})$$

with $g_2 := 2 (\varepsilon^{1/2} T^{1/2} + d^{1/2} B_0 T) \kappa_5 A_0 M_2 c^2 d^2 B_1 A_k T$. Together, (6.2) and (6.5) show the assertion (6.1). \hfill \Box

The previous theorem imposes $H^2$-regularity conditions on the data $u_0$ and $f$ which often cannot realistically be assumed in practice. Therefore, in the ensuing theorem we derive suboptimal error estimates under weaker assumptions about $u_0$ and $f$ by an interpolation argument. From now on, let $H^0 := L^2$ and (cf. Lemma 5.1)

$$H_{2\theta} := \begin{cases} H^{2\theta}, & 0 \leq \theta < 1/4, \\ H^0, & 1/4 \leq \theta \leq 1/2, \\ H^{2\theta} \cap H^0, & 1/2 < \theta \leq 1. \end{cases}$$
Lemma 6.2. Suppose conditions (A1), (B1), and (B2) are satisfied. Assume $u_0 \in H_{2\theta}$, $f \in C(J; H_{2\theta})$ for some $0 \leq \theta \leq 1$, $\theta \neq 1/4$, and $u_0 = P_h u_0$. Let \( \{u^n\}_{n=1}^N \subset H^2 \cap H^1_0 \) be the solution of (1.4) and \( \{u_h^n\}_{n=1}^N \subset H_h \) of (1.5). Then for $n = 1, \ldots, N$ there holds

\[
\|u^n - u_h^n\| \leq C_{14}(h^{2\theta} + h^\theta)(\|u_0\|_{2\theta} + \|f\|_{C(J; H^{2\theta})}),
\]

where $C_{14} := 2^{1-\theta}C_{13}^\theta C_{12}$ with $C_{13}$ being defined in Lemma 6.1. The constant $C_{13}$ with $C_{13} := 1$ for $\theta = 0$ and $\theta = 1$ does not depend on $\varepsilon$ and arises solely from the norm equivalence in Lemmas 5.1 and 5.2.

Proof. Formula (5.9) and its discrete analogue, formally obtained by substituting $A$ with its discrete approximation $A_h : H_h \rightarrow H_h$ and $E_J$ with $P_h E_J$ as well as $u_0$ and $f$ with $P_h u_0$ and $P_h f$, respectively, imply that

\[
u^n - u_h^n = F_h^n u_0 + G_h^n f
\]

with some linear operators $F_h^n$ and $G_h^n$. First, let $f \equiv 0$. Then, the $L^2$ a priori estimates (see Lemmas 5.3 and 5.4) and the error estimate (6.1) yield

\[
\|F_h u_0\| \leq 2 \|u_0\| \quad \text{for } u_0 \in L^2,
\]

\[
\|F_h^n u_0\| \leq C_{13}(h^2 + h) \|u_0\|_2 \quad \text{for } u_0 \in H^2 \cap H^1.
\]

Hence, for $0 < \theta < 1$, with $\theta \neq 1/4$, we deduce by the interpolation theorem (see [23, p. 232]) together with Lemma 5.1 that for $u_0 \in H_{2\theta}$,

\[
\|u^n - u_h^n\| = \|F_h^n u_0\| \leq 2^{1-\theta}C_{13}^\theta C_{12}(h^{2\theta} + h^\theta)\|u_0\|_{2\theta}.
\]

Conversely, let $u_0 = 0$. Arguing as before and using Lemma 5.2 instead of 5.1 gives

\[
\|u^n - u_h^n\| = \|G_h^n f\| \leq 2^{1-\theta}T^{1-\theta}C_{13}^\theta C_{12}(h^{2\theta} + h^\theta)\|f\|_{C(J; H^{2\theta})}
\]

for $f \in C(J; H_{2\theta})$. From (6.7) and (6.8) we conclude (6.6) by superposition.

Lemma 6.3. Suppose conditions (A1), (B1), and (B2) are satisfied. Assume $u_0 \in H^2 \cap H^1_0$, $f \in C(J; H^2 \cap H^1_0)$, and $u_0 = P_h u_0$. Let \( \{u^n\}_{n=1}^N \subset H^2 \cap H^1_0 \) be the solution of (1.4) and \( \{u_h^n\}_{n=1}^N \subset H_h \) of (1.5). Then for $n = 1, \ldots, N$ there holds

\[
\|u^n - u_h^n\| \leq C_{15}(h^2 + h^2k^{-1})(\|u_0\|_2 + \|f\|_{C(J; H^2)}),
\]

where $C_{15} := 2 \kappa_6 A_k M_2 e^{2d^2B_k A_k T}$ with $B_k, A_k, \kappa_6$, and $M_2$ being defined in (2.11), (2.16), (5.5), and Lemma 3.3, respectively.

Proof. Let the functions \( \{w^n_h\}_{n=1}^N \subset H_h \) satisfy the auxiliary equations

\[
k^{-1}(w^n_h - w^n_{n-1} \circ X_{n-1}, \phi_h) + \xi \langle \nabla w^n_h, \nabla \phi_h \rangle = \langle f(t_n), \phi_h \rangle
\]

for all $\phi_h \in H_h$ and $w_0^h := P_h u_0$. Then the error $u^n - u_h^n$ is expressible as

\[
u^n - u_h^n = (u^n - R_h u^n) + (R_h u^n - w_h^n) + (w_h^n - u_h^n)
\]

with $R_h$ being defined in (5.4). By (5.5) we have

\[
\|u^n - R_h u^n\| + h \|\nabla(u^n - R_h u^n)\| \leq \kappa_6 h^2 \|u^n\|_2.
\]
Next, we consider $R_h u^n - w^n_h$. We combine (1.4) with (6.10) to find

$$\langle R_h u^n - w^n_h, \phi_h \rangle + k \varepsilon \langle \nabla (R_h u^n - w^n_h), \nabla \phi_h \rangle = \langle R_h u^n - u^n, \phi_h \rangle.$$  

Putting $\phi_h = R_h u^n - w^n_h$ immediately yields

$$\|R_h u^n - w^n_h\|^2 + k \varepsilon \|\nabla (R_h u^n - w^n_h)\|^2 \leq \|u^n - R_h u^n\|^2.$$  

Hence, by (6.11) we have

$$\|R_h u^n - w^n_h\| \leq \kappa_6 h^2 \|u^n\|_2.$$  

To estimate $w^n_h - u^n_h$, we combine (6.10) with (1.5) and put $\phi_h = w^n_h - u^n_h$. Thus,

$$\|w^n_h - u^n_h\|^2 + k \varepsilon \|\nabla (w^n_h - u^n_h)\|^2 = \langle (u^{j-1} - u_h^{j-1}) \circ X^j(t_{j-1}), w^n_h - u^n_h \rangle$$  

follows. Recalling (2.4) and using elementary inequalities we obtain

$$\|w^n_h - u^n_h\| \leq \|u^{j-1} - u_h^{j-1}\| \leq \|u^{j-1} - w_h^{j-1}\| + \|w_h^{j-1} - u_h^{j-1}\|.$$  

By (6.12) and (6.13) we then deduce

$$\|w^n_h - u^n_h\| \leq 2 \kappa_6 h^2 \|u^{j-1}\|_2 + \|w_h^{j-1} - u_h^{j-1}\|.$$  

Now, we sum up (6.14). Since $w^n_h = u^n_h$, we get

$$\|w^n_h - u^n_h\| \leq 2 \kappa_6 h^2 \|u^{j-1}\|_2 \max_{0 \leq j \leq n-1} \|u^j\|_2.$$  

Finally, combining (6.11) to (6.15) with (5.8) proves the assertion (6.9). \qed

From Lemma 6.3 we deduce by the interpolation argument of Lemma 6.2.

**Lemma 6.4.** Suppose conditions (A1), (B1), and (B2) are satisfied. Assume further $u_0 \in H^{2\theta}$, $f \in C(\bar{J}; H^{2\theta})$ for some $0 \leq \theta \leq 1$, $\theta \neq 1/4$, and $u_h^n = P_h u_0$. Let $\{u^n\}_{n=1}^N \subset H^2 \cap H^1_0$ be the solution of (1.4) and $\{u^n_h\}_{n=1}^N \subset H_h$ be the solution of (1.5). Then for $n = 1, \ldots, N$ there holds

$$\|u^n - u_h^n\| \leq 2 \kappa_6 h^2 \|u^{j-1}\|_2 + \|f\|_{C(\bar{J}; H^{2\theta})},$$  

where $\kappa_6 := 2^{1-\theta} C_{15} \theta \max\{1, T^{1-\theta}\}$ with $C_{15}$ being defined in Lemma 6.3. Again, $C_{15}$ does not depend on $\varepsilon$ and arises from the norm equivalence in Lemmas 5.1 and 5.2.

Together, Lemmas 4.1, 4.2, and 4.6 as well as Lemmas 6.2 and 6.4 show the following theorem.

**Theorem 6.5.** Suppose conditions (A1), (A2), (B1), and (B2) are satisfied. Assume $u_0 \in H^{2\theta}$, $f \in C(\bar{J}; H^{2\theta})$ for some $0 \leq \theta \leq 1$, $\theta \neq 1/4$, and $u_h^n = P_h u_0$. Let $u$ be the solution of (1.1), (1.2) according to Theorem 3.1 and $\{u^n_h\}_{n=1}^N \subset H_h$ of (1.5). Then for $n = 1, \ldots, N$ there holds

$$\|u(t_n) - u_h^n\| \leq C_{12} t_n^{-1} k \left( \|u_0\|_2 + \|f\|_{C(\bar{J}; L^2)} + \|f\|_{C(\bar{J}; H^{2\theta})} \right)$$

$$+ C_{17} \left( h^{2\theta} + \min\{h^\theta, h^{2\theta}/k^\theta\} \right) \left( \|u_0\|_2 + \|f\|_{C(\bar{J}; H^{2\theta})} \right),$$  

where $C_{17} := \max\{C_{14}, C_{16}\}$ with $C_{12}$, $C_{14}$, and $C_{16}$ being defined in Lemmas 4.6, 6.2, and 6.4, respectively. If $u_0 \in H^2 \cap H^1_0$, then we have

$$\|u(t_n) - u_h^n\| \leq C_6 k \left( \|u_0\|_2 + \|f\|_{C(\bar{J}; L^2)} + \|f\|_{C(\bar{J}; H^{2\theta})} \right)$$

$$+ C_{17} \left( h^{2\theta} + \min\{h^\theta, h^{2\theta}/k^\theta\} \right) \left( \|u_0\|_2 + \|f\|_{C(\bar{J}; H^{2\theta})} \right),$$  

where $C_{17} := \max\{C_{14}, C_{16}\}$ with $C_{12}$, $C_{14}$, and $C_{16}$ being defined in Lemmas 4.6, 6.2, and 6.4, respectively. If $u_0 \in H^2 \cap H^1_0$, then we have

$$\|u(t_n) - u_h^n\| \leq C_6 k \left( \|u_0\|_2 + \|f\|_{C(\bar{J}; L^2)} + \|f\|_{C(\bar{J}; H^{2\theta})} \right)$$

$$+ C_{17} \left( h^{2\theta} + \min\{h^\theta, h^{2\theta}/k^\theta\} \right) \left( \|u_0\|_2 + \|f\|_{C(\bar{J}; H^{2\theta})} \right),$$  

where $C_{17} := \max\{C_{14}, C_{16}\}$ with $C_{12}$, $C_{14}$, and $C_{16}$ being defined in Lemmas 4.6, 6.2, and 6.4, respectively. If $u_0 \in H^2 \cap H^1_0$, then we have

$$\|u(t_n) - u_h^n\| \leq C_6 k \left( \|u_0\|_2 + \|f\|_{C(\bar{J}; L^2)} + \|f\|_{C(\bar{J}; H^{2\theta})} \right)$$

$$+ C_{17} \left( h^{2\theta} + \min\{h^\theta, h^{2\theta}/k^\theta\} \right) \left( \|u_0\|_2 + \|f\|_{C(\bar{J}; H^{2\theta})} \right),$$  

where $C_{17} := \max\{C_{14}, C_{16}\}$ with $C_{12}$, $C_{14}$, and $C_{16}$ being defined in Lemmas 4.6, 6.2, and 6.4, respectively. If $u_0 \in H^2 \cap H^1_0$, then we have

$$\|u(t_n) - u_h^n\| \leq C_6 k \left( \|u_0\|_2 + \|f\|_{C(\bar{J}; L^2)} + \|f\|_{C(\bar{J}; H^{2\theta})} \right)$$

$$+ C_{17} \left( h^{2\theta} + \min\{h^\theta, h^{2\theta}/k^\theta\} \right) \left( \|u_0\|_2 + \|f\|_{C(\bar{J}; H^{2\theta})} \right),$$  

where $C_{17} := \max\{C_{14}, C_{16}\}$ with $C_{12}$, $C_{14}$, and $C_{16}$ being defined in Lemmas 4.6, 6.2, and 6.4, respectively. If $u_0 \in H^2 \cap H^1_0$, then we have
with $C_6$ being defined in Lemma 4.1. If $u_0 \in H^2 \cap H^1_0$ and $f \equiv 0$, then

$$\|u(t_n) - u_h^n\| \leq C_{18}(\varepsilon k + h^2 + \min \{h, h^2/k\})\|u_0\|_2,$$

with $C_{18} := \max\{C_7, C_{17}\}$ and $C_7$ being defined in Lemma 4.2.

We explicitly recall that the constants $C_i$ do not depend on $0 < \varepsilon \leq 1$ and thus remain bounded in the hyperbolic limit $\varepsilon \to 0$. Further, we remark that (6.17) holds with the weaker singularity $1/t^n - \theta$ instead of $1/t_n$ if $\|u_0\|$ is replaced with $\|u_0\|_{2\theta}$. This easily follows by an interpolation argument. Inequality (6.18) even holds with $c(h^2 + \min\{h, h^2/k\}\|u_0\|_2 + c(h^{2\theta} + \min\{h^\theta, h^{2\theta}/k^\theta\}\|f\|_{C(J;H^{2\theta})})$ for the spatial discretization error. The estimate (6.19) conforms with a result proven by Morton and Süli [27] for PERU approximations of hyperbolic problems.

7. Numerical results and extensions. Finally, it remains to analyze the proven convergence rates with respect to their optimality and sharpness. For the time discretization we have proven $\varepsilon$-uniform convergence of first order for initial values in $H^2$, convergence of order $O(\varepsilon k)$ if $f \equiv 0$ additionally is supposed, and $\varepsilon$-uniform convergence of order $O(k/t_n)$ for initial values in $L^2$. This is the optimum which can be expected. We remark that the factor $\varepsilon$ in the convergence rate $O(\varepsilon k)$ can in fact be observed in numerical computations with $f \equiv 0$. Thus, the error estimate seems to be sharp. In the error estimates for the spatial discretization the term $\min\{h^\theta, h^{2\theta}/k^\theta\}$ of suboptimal order arises. We believe that under some smallness assumption about

$$P_h(E^n R_h \psi - R_h^n(t_{n-1})E^n \psi), \quad \psi \in H^1_0$$

(see definitions (2.5), (5.3), (5.4)), which is basically nothing else but the permutation of Ritz projection and change of coordinates, the term $\min\{h^\theta, h^{2\theta}/k^\theta\}$ can still be eliminated. But, it is difficult to verify the smallness of (7.1), particularly in the higher-dimensional case. Morton and Süli [27] have shown such smallness for the linear interpolant instead of the Ritz projector in one space dimension. Under a smallness condition about (7.1) the error of the spatial discretization has to be estimated through

$$\|u^n - u_h^n\| \leq \|u^n - R_h^n u^n\| + \|R_h^0 u^n - u_h^n\|,$$

and techniques as described in [3] have to be applied to $\|R_h^0 u^n - u_h^n\|$.

However, by one-dimensional computations we now illustrate the sharpness of the convergence rate $O(h^{2\theta})$ for data in $H^{2\theta}$. We consider the numerical example $\Omega = (0, 2)$, $b = 1$, $f = 0$, and $u_0$ with supp $u_0 = [0.8, 1.2]$. The final time $T = 0.1$ is chosen sufficiently small such that boundary effects are negligible. The diffusion parameter $\varepsilon$ decreases from $1.0e-05$ to $1.0e-11$. First, we compute for each choice of $\varepsilon$ a reference solution on a very fine mesh with $k = 4.0e-04$ and $h = 1.0e-05$. Then we analyze the $\varepsilon$-uniform convergence behavior of the spatial discretization by measuring the $L^2$ error between the approximate and the reference solution. The approximate solutions are computed with time step size $k = 5.0e-04$. In the numerical computations we use exact characteristic lines and evaluate all integrals exactly. The initial value $u_0$ is chosen in $H^2$ (case I) in the form of a sufficiently smooth, piecewise fourth order polynomial, in $H^{(3/2) - \delta}$ (case II) for $\delta \in (0, 3/2]$ by a triangle function, and in $H^{(1/2) - \delta}$ (case III) for $\delta \in (0, 1/2]$ by a square function. We use least squares fitting to obtain the order of convergence. The numerical results are contained in Table 7.1. They clearly indicate $\varepsilon$-uniform convergence of second order for $u_0 \in H^2$, $\varepsilon$-uniform convergence of nearly order 3/2 for $u_0 \in H^{(3/2) - \delta}$ and, finally, $\varepsilon$-uniform convergence of order 1/2 for
Test for ε-uniform convergence rates of the spatial discretization.

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Convergence order 2.085 1.374 0.512

$u_0 \in H^{1/2-\delta}$. Hence, the convergence rate $O(h^{2\delta})$ for data in $H^{2\delta}$ seems to be sharp. In our one-dimensional computations we could not observe the term $\min \{h^{\delta}, h^{2\delta}/k^{\delta} \}$.

Summarizing, the Lagrangian framework which we have developed seems to be the adequate technique for analyzing characteristic-based approximations of convection-dominated problems. We have established a theoretical basis for the Lagrange–Galerkin approach. We believe that our fundamental techniques can be applied or extended to a much larger class of problems and to other characteristic-based approximation schemes, e.g., schemes of ELLAM type (see, e.g., [34]). In current analyses of these methods, the error constants also depend on higher order norms of the solution; see [34]. To estimate these terms, our Lagrangian techniques might be applicable.

In this paper we have not addressed the matter of approximate characteristic lines. If (1.3) is discretized by, e.g., the explicit Euler scheme, then a smallness condition has to be imposed on the time step size in order to ensure that the resulting approximate characteristics define a diffeomorphism of $\bar{\Omega}$ onto itself; see [29, p. 94]. It seems possible to prove now similar error estimates. But the error has to be split somewhat stronger. However, our estimates show which uniform convergence rates can realistically and at most be expected under certain regularity assumptions about the data.

An extension of our analysis to problems with matrix-valued diffusion terms is straightforward. For instance, in the case of a constant, symmetric, and positive definite diffusion matrix $D$, the error estimates hold uniformly with respect to $\lambda_{\max}/\lambda_{\min}$, where $\lambda_{\max}$ and $\lambda_{\min}$ are the largest and smallest eigenvalue of $D$. If $D$ varies in time and space, then $\lambda_{\max}$ has to be replaced by the maximum of some norm of the matrix elements $D_{ij} = D_{ij}(t, x)$ and their time derivatives, respectively. The norms indicated by (2.11) arise. Our techniques can also be applied to convection-diffusion-reaction problems. Even some classes of nonlinear reaction terms are admissible. We will show this in a forthcoming paper, but the analysis becomes more difficult and technical. For instance, a bound for $u^n$ in $C(J_n; H^2)$ can be established by using the semigroup approach of Lemma 3.9. The reaction term is written on the right-hand side of the evolution equation. Now, a priori estimates in weaker, also non–Hilbert space norms (e.g., in $L^\infty$) have to be established first and an additional Gronwall argument has to be used. In this regard we note that our semigroup techniques work analogously in $L^p$ spaces.

In practice many convection-diffusion problems are considered in domains with inflow and outflow boundaries. As far as we know, no theoretical analysis of Lagrange–Galerkin methods applied to such problems has been published yet. We believe that our techniques can be generalized to these problems by introducing an appropriate extension of the convection-diffusion problem beyond the boundary. Then the results
of this paper might be applicable in the extended domain. Moreover, such rigorous analysis will provide more information about the proper choice of the inflow and outflow boundary conditions for Lagrange–Galerkin methods and the extension of the solution at the old time level beyond the boundary. This work is still in process.

If $b$ is not divergence-free ($\nabla \cdot b \neq 0$), then instead of (3.2) we have

$$g^n \partial_t \bar{u}^n - \nabla \cdot (g^n D^n \nabla \bar{u}^n) = g^n \bar{f}^n$$

with $D^n$ defined as in (2.10) and $g^n(t, x) := |\det \nabla_x X(t; t_n, x)|$. Under appropriate assumptions about $b$ the analysis can now be done in a similar way but becomes more technical. The basic distinction is that an additional “weight function” $g^n$ satisfying $g^n(t_n, x) = 1$ has to be taken into account in the transformation rules, in identities (2.3) and (2.4), and in the inner products arising in the a priori and error analysis.

**Acknowledgments.** The authors wish to thank the referees for their comments and suggestions for improving the presentation of the paper.

**REFERENCES**


