FINITE ELEMENT APPROXIMATION OF THE TRANSPORT OF REACTIVE SOLUTES IN POROUS MEDIA. PART II: ERROR ESTIMATES FOR EQUILIBRIUM ADSORPTION PROCESSES

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Abstract. In this paper we analyze a fully practical piecewise linear finite element approximation involving numerical integration, backward Euler time discretization, and possibly regularization and relaxation of the following degenerate parabolic equation arising in a model of reactive solute transport in porous media: find $u(x,t)$ such that

$$\partial_t u + \partial_t [\varphi(u)] - \Delta u = f \quad \text{in } \Omega \times (0,T],$$

$$u = 0 \quad \text{on } \partial \Omega \times (0,T] \quad u(\cdot,0) = g(\cdot) \quad \text{in } \Omega$$

for known data $\Omega \subset \mathbb{R}^d$, $1 \leq d \leq 3$, $f$, $g$, and a monotonically increasing $\varphi \in C^0(\mathbb{R}) \cap C^1(-\infty,0] \cup (0,\infty)$ satisfying $\varphi(0) = 0$, which is only locally Hölder continuous with exponent $p \in (0,1)$ at the origin; e.g., $\varphi(s) \approx |s|^p$. This lack of Lipschitz continuity at the origin limits the regularity of the unique solution $u$ and leads to difficulties in the finite element error analysis.

Key words. finite elements, error analysis, degenerate parabolic problem, porous medium

AMS subject classifications. 65M15, 65M60, 35K55, 35K65, 35R35

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1. Introduction. This is the second of two papers in which we study finite element approximations of degenerate parabolic systems and equations as they arise in the modeling of reactive solute transport in porous media. Here we concentrate on a quasi-stationary equilibrium adsorption reaction leading to

(1.1a) \[ \partial_t (\Theta u) + \rho \partial_t v - \nabla \cdot (\Theta D \nabla u - qu) = f \quad \text{in } Q_T, \]

(1.1b) \[ v = \varphi(u) \quad \text{in } Q_T, \]

supplemented by initial and boundary conditions for the unknown function $u$, the dissolved concentration. Here $\Omega$ is a bounded domain in $\mathbb{R}^d$, $1 \leq d \leq 3$, $[0,T]$ is a fixed time interval, and $Q_t \equiv \Omega \times (0,t]$ for $t \in (0,T]$. For more remarks on the model we refer to the introduction of Part I [3] and for a complete account to the literature cited there. The parameter functions $\Theta, \rho, D, \rho$ are given and fulfill

(1.2) \[ \partial_t \Theta + \nabla \cdot q = 0, \quad \Theta(x,t) \geq \Theta_0 > 0, \quad \rho(x) \geq \rho_0 > 0 \quad \text{in } Q_T, \]

and further conditions such that the linear part of (1.1) defines a uniformly parabolic operator. The nonlinearity $\varphi$—the adsorption isotherm—is monotone nondecreasing.

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but there are typical examples which are not Lipschitz continuous at \(u = 0\) such as the Freundlich isotherm

\[
\varphi(u) \equiv \alpha u^p \text{ for } u \geq 0, \text{ where } \alpha \in \mathbb{R}^+ \text{ and } p \in (0, 1).
\]

Thus in general equation (1.1) is degenerate, exhibiting a finite speed of propagation property, such that a front given by the boundary of the support of \(u\) is preserved. In fact, there is a close relation between equation (1.1) and the well-investigated (generalized) “porous medium equation” (see, e.g., [1]) which reads

\[
\partial_t \varphi(u) - \Delta u = f \quad \text{in } Q_T
\]

with \(\varphi(u) \equiv \text{sgn}(u)|u|^{1/m}\) for some \(m > 1\); i.e., (a model problem of) (1.1) is of the form (1.4a), and (1.1) and (1.4a) are equivalent if we assume that for some \(\alpha > 0\)

\[
\varphi'(u) \geq \alpha \quad \forall u \in \mathbb{R}, \quad u \neq 0.
\]

A sufficient condition for the finite speed of propagation property is

\[
1/\phi \in L^1(0, \delta) \quad \text{for some } \delta > 0;
\]

see [16], which has also been proven to be necessary in the one-dimensional case, and see [12]. This condition is satisfied by (1.3); see also section 2. A common description of chemical nonequilibrium has the form of relaxation kinetics; i.e.,

\[
\partial_t v = k(\varphi(u) - v)
\]

with a rate parameter \(k > 0\). Equations (1.1a), (1.6) in general form a degenerate system with the aforementioned property. In Part I we gave a fairly complete order of convergence analysis in energy norms for Galerkin finite element approximations of the system (1.1a), (1.6); based on a technique which is at least applicable for time-independent and smooth \(\Theta, q, D\). However, the fact that we analyzed the Galerkin procedure assumes in addition that the system is not convection dominated, where we would encounter all the well-known difficulties. This analysis has been presented for a model problem, to which we will restrict ourselves later on.

One may expect that for \(k \to \infty\) \((P_k) \equiv (1.1a), (1.6)\) approximates \((P) \equiv (1.1)\). This may be called a kinetic approximation and will be made rigorous in section 2. The aim of this paper is to exploit the kinetic approximation as a proof technique device (and possibly also as an algorithmic device) to study order of convergence estimates for the problem \((P)\) on the basis of the results of Part I for the relaxed problem \((P_k)\). There it turned out to be advantageous to introduce a regularized system \((P_{k,\varepsilon})\) obtained by substituting \(\varphi\) by a Lipschitz continuous \(\varphi_{\varepsilon}\), differing only near \(u = 0\). The relaxation is a proof device insofar as the order of convergence estimates, established for the finite element approximation of \((P_{k,\varepsilon})\) for appropriate \(k = O(h^{-\gamma}), \varepsilon = O(h^\mu)\), where \(h\) is the mesh parameter, then carry over to the corresponding finite element approximation of \((P_\varepsilon)\), the regularized version of \((P)\). We can improve on these convergence estimates by taking into account a nondegeneracy (N.D.) condition, which describes the minimal growth of \(u\) away from the front. In the one-dimensional case the following result has been established in [2]. We will assume later on that \(\varphi\) is Hölder continuous near \(u = 0\) with exponent \(p \in (0, 1)\). If in addition the exponent is sharp, i.e.,

\[
\varphi(u) \geq \alpha u^p \quad \text{for } u \in [0, \delta_0] \quad \text{and for some } \alpha, \delta_0 > 0,
\]
then
\[(1.8a) \quad A_\epsilon(t) \leq C \epsilon, \quad (\text{N.D.})\]
where
\[(1.8b) \quad A_\epsilon(t) \equiv \int_0^t m(\Omega_\epsilon(s)) \, ds,\]
\[(1.8c) \quad \Omega_\epsilon(t) \equiv \{ x \in \Omega : u(x,t) \in (0, \epsilon^{1/(1-p)}) \},\]
and \(m\) is the Lebesgue measure.

For ease of exposition we will develop our results for the following model problem, which keeps the specific difficulty of the non-Lipschitz nonlinearity but reduces the handling of standard terms:

(P) Find \(u(x,t)\) such that
\[
\partial_t u + \partial_t [\varphi(u)] - \Delta u = f \quad \text{in } Q_T,
\]
\[
u = 0 \quad \text{on } \partial \Omega \times (0,T], \quad u(\cdot,0) = g(\cdot) \quad \text{in } \Omega,
\]
where we make the following assumptions on the given data.

(D1). \(\Omega \subset \mathbb{R}^d, 1 \leq d \leq 3\), with either \(\Omega\) convex polyhedral or \(\partial \Omega \in C^{1,1}\), \(f \in L^\infty(Q_T), g \in L^\infty(\Omega) \cap H^1_0(\Omega), \) and \(\varphi \in C^0(\mathbb{R})\) is such that
\[(1.9a) \quad \varphi(0) = 0, \varphi(s) > 0 \text{ for all } s > 0 \text{ and } \varphi \text{ is monotonically increasing},\]
\[(1.9b) \quad \varphi \in C^1(-\infty,0] \cup [0,\infty),\]
there exist \(L \in \mathbb{R}^+\) and \(\varepsilon_0, p \in (0,1]\) such that
\[(1.9c) \quad |\varphi(a) - \varphi(b)| \leq L|a - b|^p \quad \forall a, b \in [0,\varepsilon_0].\]

Below we gather the further assumptions that we will require at various stages in the paper.

(D2). In addition to (D1) we assume that \(f \in H^1(0,T; L^2(\Omega)), g \in H^2(\Omega)\) and that \(k \geq k_0\).

(D3). In addition to (D2) we assume that the constant \(M\) in (2.2c) can be chosen uniformly for all \(s \in \mathbb{R}\). (In view of the bounds (2.5) for \(u\), see Theorem 2.2, this is always achievable by changing \(\varphi(s)\) for \(|s| \geq m = \max\{-\bar{u}, \bar{v}\}\).) Let \(\Omega^h\) be a polyhedral approximation to \(\Omega\) defined by \(\Omega^h \equiv \cup_{\kappa \in T^h} \kappa\) with \(\text{dist}(\partial \Omega, \partial \Omega^h) \leq Ch^2\), where \(T^h\) is a partitioning consisting of regular simplices \(\kappa\) with \(h_{\kappa} \equiv \text{diam}(\kappa)\) and \(h \equiv \max_{\kappa \in T^h} h_{\kappa}\). For ease of exposition we assume that \(\Omega^h \subseteq \Omega\).

(D4). In addition to (D3) we assume that \(f \in H^1(0,T; C^0(\Omega)) \cap L^2(0,T; H^2(\Omega))\).

We prove our basic error bound for a fully practical approximation to (P) under assumption (D5).

(D5). In addition to (D4) we assume that \(T^h\) is such that
(i) for \(d = 2\) it is weakly acute; that is, for any pair of adjacent triangles the sum of opposite angles relative to the common side does not exceed \(\pi\);
(ii) for \(d = 3\) the angle between any two faces of the same tetrahedron does not exceed \(\pi/2\).

We improve on these basic error bounds by replacing (D5) by (D6).
(D6). In addition to (D5) we assume that
(i) $\Omega \subset \mathbb{R}^d$, $d = 1$ or $2$, and $T^h$ is a quasi-uniform partition if $d = 2$;
(ii) $g$ and $f \geq 0$;
(iii) $\varphi \in C^2(0, m]$ such that $\varphi''(s) \leq 0$ for all $s \in (0, m]$, where $m = \max\{\|\varphi\|\}$; see (2.5).

We note that in proving the error bounds in this paper, as opposed to the bounds in, for example, Corollary 3.1 and (4.1), the only place that the acuteness assumption on the partitioning $T^h$ is required is in establishing the bounds (3.7) and (3.8).

The layout of this paper is as follows. In the next section we establish the existence and uniqueness of a solution to (P) under assumption (D1) by first establishing these results for a regularized relaxed version $(P_{k,\varepsilon})$. In addition we recall a number of useful a priori estimates for $(P_{k,\varepsilon})$ under assumptions (D1) and (D2). In section 3 under assumption (D5) we prove error bounds for a continuous in time, continuous piecewise linear finite element approximation in space involving numerical integration of $(P_{\varepsilon})$. Moreover, we improve on these bounds under assumption (D6). In addition we note that one can prove superior error bounds for a less practical scheme involving no numerical integration under assumption (D3). In section 4 we consider a fully practical approximation involving discretization in time using the backward Euler method. Finally, in section 5 we report on a numerical experiment.

The most complete order of convergence analysis until now for the finite element approximation of the porous medium equation involving time discretization and numerical integration has been given in [11]. In fact many of the proof techniques used in this paper are similar to those used there. However, contrary to our approach, they consider this approximation directly, taking regularization but not relaxation of the problem into account. A proviso in the comparison lies in the fact that in some places we require the mesh to be (weakly) acute, whereas they do not. Our approach first leads to their resulting error bounds with a less severe time step constraint; that is, $\tau = Ch$ as opposed to their restrictions $\tau = Ch^{1+p}$ and $\tau = Ch^{4/(3-p)}$ on not assuming and assuming (N.D.), respectively. Furthermore, under additional assumptions, see (D6), we can improve on their error bounds in some cases.

Finally, we note that one could employ alternative forms of relaxation not considered here. The description of a physically caused nonequilibrium may lead to

$$\partial_t v = k(u - \varphi^{-1}(v)).$$

(1.10)

For a nonlinearity of the type (1.3), $\varphi^{-1}$ is Lipschitz continuous; i.e. (1.1a), (1.10) is a regular system. This type of relaxation was used in [15] for the Stefan problem. A semi-implicit time discretization leads to a linear elliptic problem at each time level; whereas for the relaxation (1.6) considered here, a fully implicit time discretization leads to a semilinear elliptic problem of the type studied in [9] and [5]. Therefore, from a computational viewpoint it would seem advisable to use a relaxation based on (1.10). However, we note that (1.10), as it leads to a regular system, adds a considerable amount of artificial diffusion. This has already been reported in [15]. In [7] a modification of relaxation is proposed to improve on this aspect, which leads to a nonlinear algorithm at each time level. A relaxation based on (1.6) smears much less, as it still leads to a problem with a finite speed of propagation. Therefore, for computational purposes, we have to leave it open whether relaxation, and in which form, is to be preferred. We stress again that in this paper the relaxation based on (1.6) is mainly used as a proof technique to establish order of convergence results for the finite element approximation of (P) (without relaxation), which we are unable to prove otherwise. As an intermediate step, we obtain the same results for
an approximation with relaxation. Here the computational complexities of the fully
 discrete problems are identical.
 Throughout the paper we adopt the standard notation for Sobolev spaces. We
 note that the seminorm $|\cdot|_{H^1(\Omega)}$ and the norm $\|\cdot\|_{H^1(\Omega)}$ are equivalent on $H^1_0(\Omega)$. The
 standard $L^2$ inner product over $\Omega$ is denoted by $(\cdot, \cdot)$. Throughout $C$ denotes a generic
 positive constant independent of $\varepsilon$ the regularization parameter, $k$ the relaxation
 parameter, and $h$ the mesh spacing.

2. The continuous problem. In this section we review the existence and uni-
 queness of a solution to (P). In doing so we will develop various bounds that will
 be useful in analyzing the error in the finite element approximation of (P). First we
 introduce a regularized version of (P) for $\varepsilon \in (0, \varepsilon_0]$ ($\varepsilon_0$ as in (1.9b)):

(P$_{\varepsilon}$) Find $u_{\varepsilon}(x,t)$ such that

$$
\partial_t u_{\varepsilon} + \partial_t[\varphi_{\varepsilon}(u_{\varepsilon})] - \Delta u_{\varepsilon} = f \quad \text{in } Q_T,
$$

$$
u_{\varepsilon} = 0 \quad \text{on } \partial \Omega \times (0, T], \quad u_{\varepsilon}(\cdot, 0) = g_{\varepsilon}(\cdot) \quad \text{in } \Omega,
$$

where $g_{\varepsilon} \in L^\infty(\Omega) \cap H^1_0(\Omega)$ is such that

$$
(2.1a) \quad g_{\varepsilon} \to g \text{ in } L^\infty(\Omega) \cap H^1_0(\Omega) \text{ as } \varepsilon \to 0,
$$

$$
(2.1b) \quad |g_{\varepsilon}|_{H^2(\Omega)} \leq C \text{ if } g \in H^2(\Omega),
$$

and $\varphi_{\varepsilon} \in C^0_{\text{loc}}(\mathbb{R})$ is such that

$$
(2.2a) \quad \varphi_{\varepsilon}(s) \equiv \varphi(s) \text{ for } s \notin (0, \varepsilon^{1/(1-p)}),
$$

$$
(2.2b) \quad \varphi_{\varepsilon}(s) \text{ is strictly monotonically increasing on } [0, \varepsilon^{1/(1-p)}],
$$

for $m \in \mathbb{N}$ there exists a $M(m) \in \mathbb{R}^+$:

$$
(2.2c) \quad \varphi_{\varepsilon}(b) - \varphi_{\varepsilon}(a) \leq M(m)\varepsilon^{-1}(b-a) \quad \text{for } -m \leq a \leq b \leq m.
$$

Note that $M$ can be chosen independently of $m$ if $\varphi'$ is bounded in $\mathbb{R} \setminus (0, \delta)$ for some
$\delta > 0$. In addition we set

$$
\Phi_{\varepsilon}(s) = \int_0^s \varphi_{\varepsilon}(\sigma) \, d\sigma.
$$

It is a simple matter to deduce from the conditions (2.2) that for all $|a|, |b| \leq m$,

$$
(2.3a) \quad [M(m)]^{-1} \varepsilon |\varphi_{\varepsilon}(a) - \varphi_{\varepsilon}(b)|^2 \leq [\varphi_{\varepsilon}(a) - \varphi_{\varepsilon}(b)](a - b) \leq M(m)\varepsilon^{-1}|a - b|^2
$$

and

$$
(2.3b) \quad \varphi_{\varepsilon}(\varepsilon^{1/(1-p)}) = \varphi(\varepsilon^{1/(1-p)}) \leq L\varepsilon^{p/(1-p)}
$$

with $L$ as in (1.9b). The simplest choice for $\varphi_{\varepsilon}$ is the linear regularization

$$
(2.4) \quad \varphi_{\varepsilon}(s) \equiv \varepsilon^{-1/(1-p)}\varphi(\varepsilon^{1/(1-p)}) \quad \text{for } s \in (0, \varepsilon^{1/(1-p)}).
$$

In addition it is useful to consider the following problem, in which the reaction
 process is relaxed in time with $k > 0$ being the given relaxation parameter.

(P$_{k,\varepsilon}$) Find $(u_{k,\varepsilon}(x,t), v_{k,\varepsilon}(x,t))$ such that

$$
\partial_t u_{k,\varepsilon} + \partial_t v_{k,\varepsilon} - \Delta u_{k,\varepsilon} = f \quad \text{in } Q_T, \quad u_{k,\varepsilon} = 0 \quad \text{on } \partial \Omega \times (0, T],
$$

$$
\partial_t v_{k,\varepsilon} = k(\varphi_{\varepsilon}(u_{k,\varepsilon}) - v_{k,\varepsilon}) \quad \text{in } Q_T,
$$

$$
u_{k,\varepsilon}(\cdot, 0) = g_{\varepsilon}(\cdot), \quad v_{k,\varepsilon}(\cdot, 0) = \varphi_{\varepsilon}(g_{\varepsilon}(\cdot)) \quad \text{in } \Omega.
$$
The above problem has been studied in Part I [3]. We adopt the notion of a weak solution defined there and below we recall some of the results.

**THEOREM 2.1.** Let assumption (D1) hold. Then for all \( \varepsilon \in (0, \varepsilon_0] \) and \( k > 0 \) there exists a unique weak solution \( \{ u_{k,\varepsilon}, v_{k,\varepsilon} \} \) to \((P_{k,\varepsilon})\) such that

\[
(2.5) \quad u \leq u_{k,\varepsilon} \leq \overline{u} \quad \text{and} \quad v \leq v_{k,\varepsilon} \leq \overline{v} \quad \text{in} \quad Q_T,
\]

where \( u, \overline{u}, v, \overline{v} \in C(\overline{\Omega}) \) are all independent of \( \varepsilon \) and \( k \). Furthermore, if \( g_{\varepsilon} \) and \( f \geq 0 \) one can take \( u = v = 0 \).

**Proof.** This result with \( u, \overline{u}, v, \overline{v} \in C[0, T] \), all independent of \( \varepsilon \) and uniformly bounded in \( k \), follows from Theorem 2.1 of Part I [3]. Furthermore noting Remark 2.1 of Part I yields the above choice of \( u, \overline{u}, v, \overline{v} \). We note for later purposes that \( u \) and \( \overline{u} \) depend only on \( \Omega \), \( |f|_{L^\infty(Q_T)} \) and \( |g_{\varepsilon}|_{L^\infty(\Omega)} \).

**LEMMA 2.1.** Under assumption (D1) we have for all \( \varepsilon \in (0, \varepsilon_0], k > 0 \), and \( t \in (0, T] \) that

\[
(2.6) \quad \varepsilon |\nabla \varphi_{\varepsilon}(u_{k,\varepsilon})|_{L^2(Q_T)}^2 + \int_0^T \langle \nabla u_{k,\varepsilon}(\cdot, s), \nabla \varphi_{\varepsilon}(u_{k,\varepsilon}(\cdot, s)) \rangle \, ds + \langle \Phi_{\varepsilon}(u_{k,\varepsilon}(\cdot, t)), 1 \rangle + k \varepsilon |\nabla \varphi_{\varepsilon}(u_{k,\varepsilon})|_{L^2(Q_T)}^2 + |v_{k,\varepsilon}(\cdot, t)|_{L^2(\Omega)}^2 + k^{-1}|\partial_t v_{k,\varepsilon}|_{L^2(Q_T)}^2 \leq C.
\]

**Proof.** The proof is provided in Lemma 2.1 of Part I [3].

**LEMMA 2.2.** Let assumption (D2) hold. Then we have for all \( \varepsilon \in (0, \varepsilon_0], k > 0 \), and \( t \in (0, T] \) that

\[
|\nabla u_{k,\varepsilon}(\cdot, t)|_{L^2(\Omega)}^2 + |\partial_t u_{k,\varepsilon}|_{L^2(Q_T)}^2 + \varepsilon|\partial_t v_{k,\varepsilon}|_{L^2(Q_T)}^2 + |v_{k,\varepsilon}(\cdot, t)|_{L^2(\Omega)}^2 + \varepsilon|\partial_t v_{k,\varepsilon}(\cdot, t)|_{L^2(\Omega)}^2 + \left| \nabla (\partial_t u_{k,\varepsilon}(\cdot, t)) \right|_{L^2(Q_T)}^2 \leq C.
\]

**Proof.** Noting (2.1b), see Lemma 2.2 in Part I [3].

We will prove existence of solutions of problems \((P_{\varepsilon})\) or \((P)\) in the following sense.

**DEFINITION.** \( u_{\varepsilon} \) is a weak solution to \((P_{\varepsilon})\) if \( u_{\varepsilon} \in L^2(0, T; H^1_0(\Omega)) \) and for all test functions \( \eta \in L^2(0, T; H^1_0(\Omega)) \) with \( \eta(\cdot, 0) = 0 \) in \( \Omega \)

\[
\int_{Q_T} \{ -[u_{\varepsilon} + \varphi_{\varepsilon}(u_{\varepsilon})] \partial_t \eta + \nabla u_{\varepsilon} \cdot \nabla \eta - f \eta \} \, dx \, dt - \langle |g_{\varepsilon}(\cdot)| + \varphi_{\varepsilon}(g_{\varepsilon}(\cdot)), \eta(\cdot, 0) \rangle = 0.
\]

A similar definition holds for \((P)\) with \( u_{\varepsilon}, \varphi_{\varepsilon}(u_{\varepsilon}), \) and \( g_{\varepsilon} \) replaced by \( u, \varphi(u), \) and \( g \).

For \( k \in \mathbb{R}_+^* \equiv \mathbb{R}^+ \cup \{ \infty \} \) and for sufficiently smooth \( w \) we set

\[
\|w\|_{E_1(k,t)}^2 \equiv \|w\|_{L^2(Q_r)}^2 + \frac{1}{2} k^{-1} \int_0^t |w(\cdot, s)|_{L^2(\Omega)}^2 \, ds
\]

and

\[
\|w\|_{E_2(k,t)}^2 \equiv \|w\|_{E_1(k,t)}^2 + \frac{1}{2} \left\| \nabla \right\|_{L^2(\Omega)}^2 \int_0^t |w(\cdot, s)|_{L^2(\Omega)}^2 \, ds + k^{-1} \left\| \nabla w \right\|_{L^2(Q_r)}^2.
\]

**THEOREM 2.2.** Let assumption (D1) hold. Then for all \( \varepsilon \in (0, \varepsilon_0] \) there exists a unique weak solution \( u_{\varepsilon} \) to \((P_{\varepsilon})\) and

\[
(2.8a) \quad u \leq u_{\varepsilon} \leq \overline{u} \quad \text{in} \quad Q_T,
\]
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where \( u, \pi \in C^0(\Omega) \) are all independent of \( \varepsilon \). Moreover, if \( g_\varepsilon \) and \( f \) \( \geq 0 \) then \( u_\varepsilon \geq 0 \) in \( Q_T \). Furthermore, for all \( k > 0 \) and \( t \in (0, T) \) we have that

\[
|u_\varepsilon - u_{k,\varepsilon}|^2_{E^2(\infty, t)} + \varepsilon |\varphi_\varepsilon(u_\varepsilon) - \varphi_\varepsilon(u_{k,\varepsilon})|^2_{L^2(Q_t)} + \varepsilon |\varphi_\varepsilon(u_\varepsilon) - v_{k,\varepsilon}|^2_{L^2(Q_t)}
\]

(2.8b)

\[
\leq C k^{-2} |\partial_t v_{k,\varepsilon}|^2_{L^2(Q_T)} \leq \begin{cases} 
Ck^{-1} & \text{if (D1) holds}, \\
C^{-1}k^{-2} & \text{if (D2) holds}.
\end{cases}
\]

**Proof.** Existence of a weak solution \( u_\varepsilon \) to (P\(_\varepsilon\)) follows by letting \( k \to \infty \) in (P\(_{k,\varepsilon}\)), from which it is clearly seen from (2.5) that the bounds (2.8a) hold; see [4] for details. Let \( e \equiv u_1^\varepsilon - u_2^\varepsilon \), where \( u_1^\varepsilon, u_2^\varepsilon \) are two weak solutions of (P\(_\varepsilon\)). Then subtracting the two defining equations and choosing \( \eta(\cdot, t) \equiv \int_t^T e(\cdot, s) \, ds \) yields \( \|e_{k,\varepsilon}\|^2_{E^2(\infty, T)} \leq 0 \) on noting (2.3a) and hence uniqueness.

Let \( e_{k,\varepsilon}^n \equiv u_\varepsilon - u_{k,\varepsilon} \) and \( e_{k,\varepsilon}^{n+1} \equiv \varphi_\varepsilon(u_\varepsilon) - v_{k,\varepsilon} \). Then subtracting the first equation in (P\(_{k,\varepsilon}\)) from (P\(_\varepsilon\)), multiplying by \( \int_s^t e_{k,\varepsilon}^n(\cdot, \sigma) \, d\sigma \), integrating over \( Q_t \), where \( s \) is the integration variable in time, and performing integration by parts and noting (2.3a) and the second equation in (P\(_{k,\varepsilon}\)) yields that

\[
\|e_{k,\varepsilon}^n\|^2_{E^2(\infty, t)} + [M(m)]^{-1} \varepsilon |\varphi_\varepsilon(u_\varepsilon) - \varphi_\varepsilon(u_{k,\varepsilon})|^2_{L^2(Q_t)} \\
\leq \|e_{k,\varepsilon}^{n+1}\|^2_{E^2(\infty, t)} + \int_0^t \left( |\varphi_\varepsilon(u_\varepsilon(\cdot, s)) - \varphi_\varepsilon(u_{k,\varepsilon}(\cdot, s))|^2 + |e_{k,\varepsilon}^n(\cdot, s)|^2 \right) \, ds
\]

(2.9)

\[
= -k^{-1} \int_0^t \left( |\partial_s v_{k,\varepsilon}(\cdot, s)|^2 + |e_{k,\varepsilon}^n(\cdot, s)|^2 \right) \, ds,
\]

where \( \inf u, \sup \pi \subseteq [-m, m] \); see Theorem 2.1. Hence the desired result (2.8b) follows from (2.9), (2.6), and (2.7). \( \square \)

**Theorem 2.3.** Let assumption (D1) hold. Then there exists a unique weak solution \( u \) to (P) and for all \( \varepsilon \in (0, \varepsilon_0) \) and \( t \in (0, T) \) we have that

\[
\|u - u_\varepsilon\|^2_{E^2(\infty, t)} + \varepsilon |\varphi(u) - \varphi_\varepsilon(u_\varepsilon)|^2_{L^2(Q_t)} \\
\leq C A_\varepsilon(t) \varepsilon^{(1+p)/(1-p)} + C[[g - g_\varepsilon] + |\varphi(g) - \varphi_\varepsilon(g_\varepsilon)|]_{H^{-1}(\Omega)}.
\]

(2.10)

In addition, the bounds (2.8a) hold for \( u \), and if \( g, f \geq 0 \) then \( u \geq 0 \) in \( Q_T \).

**Proof.** Existence of a weak solution \( u \) to (P) follows by letting \( \varepsilon \to 0 \) in (P\(_\varepsilon\)), from which it is clearly seen that the bounds (2.8a) hold for \( u \); see [4] for details. Uniqueness follows as for (P\(_\varepsilon\)) with \( \varphi_\varepsilon \) replaced by \( \varphi \); see Theorem 2.2.

Finally to show (2.10), let \( e^u \equiv u - u_\varepsilon \) and \( e^v \equiv \varphi(u) - \varphi_\varepsilon(u_\varepsilon) \). Subtracting the first equation in (P\(_\varepsilon\)) from (P), multiplying by \( \int_s^t e^u(\cdot, \sigma) \, d\sigma \), integrating over \( Q_t \) yields

\[
\|e^u\|^2_{E^2(\infty, t)} = -\int_0^t \left< e^v(\cdot, s), e^u(\cdot, s) \right> \, ds + \left< e^u(\cdot, 0), e^v(\cdot, 0) \right> + \int_0^t e^u(\cdot, s) \, ds
\]

and therefore, noting (2.3a),

\[
\|e^u\|^2_{E^2(\infty, t)} + [M(m)]^{-1} \varepsilon |e^u|^2_{L^2(Q_t)} \\
\leq \int_0^t \left< e^v(\cdot, s), (\zeta - u)(\cdot, s) \right> \, ds + \left< e^u(\cdot, 0), e^v(\cdot, 0) \right> + \int_0^t e^u(\cdot, s) \, ds
\]

\[
\leq C A_\varepsilon(t) \varepsilon^{(1+p)/(1-p)} + C[[g - g_\varepsilon] + |\varphi(g) - \varphi_\varepsilon(g_\varepsilon)|]_{H^{-1}(\Omega)},
\]
where \( \zeta \equiv \varphi^{-1}_\varepsilon(\varphi(u)) \) if \( \varphi(u) \in (0, \varphi(\varepsilon^{1/(1-p)}) \) and \( \zeta \equiv u \) otherwise, and \( \inf \Omega, \sup \Omega \subseteq [-m, m] \).

Next we discuss possible choices of \( g_\varepsilon \).

(a) On choosing \( g_\varepsilon = g \) it follows from Sobolev embedding, (2.2a), (2.2b), and (2.3b) that

\[
\| g - g_\varepsilon \| + \| \varphi(g) - \varphi_\varepsilon(g_\varepsilon) \|_{H^{-1}(\Omega)} \leq C \| \varphi(g) - \varphi_\varepsilon(g_\varepsilon) \|_{L^2(\Omega)}^2 \leq C [m(\Omega_{g,\varepsilon})]^2 \varepsilon^{2p/(1-p)},
\]

(2.11a)

where \( \Omega_{g,\varepsilon} \equiv \{ x \in \Omega : g(x) \in (0, \varepsilon^{1/(1-p)}) \} \), \( \mu = 1 \) if \( d = 1 \), \( \mu > 1 \) if \( d = 2 \), and \( \mu = 6/5 \) if \( d = 3 \). Hence if \( m(\Omega_{g,\varepsilon}) \varepsilon^{2p/(1-p)} \leq C A_\varepsilon(T) \varepsilon^{(1+p)/(1-p)} \), that is,

\[
m(\Omega_{g,\varepsilon}) \leq C[A_\varepsilon(T)\varepsilon]^{\mu/2},
\]

then it is acceptable to choose \( g_\varepsilon = g \). However, for general \( g \in H^2(\Omega) \) this choice leads to an inferior rate of convergence as \( \varepsilon \to 0 \) compared with the first term on the right-hand side of (2.10).

(b) On choosing \( g_\varepsilon \) such that \( \varphi_\varepsilon(g_\varepsilon) = \varphi(g) \) yields that \( g_\varepsilon = g \) if \( g \notin (0, \varepsilon^{1/(1-p)}) \) and hence it follows that

\[
\| g - g_\varepsilon \| + \| \varphi(g) - \varphi_\varepsilon(g_\varepsilon) \|_{H^{-1}(\Omega)} \leq C m(\Omega_{g,\varepsilon}) \varepsilon^{(1+p)/(1-p)}.
\]

However, \( g_\varepsilon \notin H^2(\Omega) \) in general and therefore many of the a priori estimates do not hold, e.g., (2.7). Below we seek an alternative choice for \( g_\varepsilon \).

**Corollary 2.1.** Let assumption (D2) hold. Then for all \( \varepsilon \in (0, \varepsilon_0] \) on choosing \( g_\varepsilon \in H^1_0(\Omega) \) to be the unique solution of

\[
\langle \nabla g_\varepsilon, \nabla \eta \rangle + \langle \varphi_\varepsilon(g_\varepsilon), \eta \rangle = \langle \nabla g, \nabla \eta \rangle + \langle \varphi(g), \eta \rangle \quad \forall \eta \in H^1_0(\Omega)
\]

yields that \( g - g_\varepsilon \|_{L^\infty(\Omega)} \leq \varepsilon^{1/(1-p)} \) and

\[
\varepsilon [g - g_\varepsilon]_{H^2(\Omega)}^2 + |g - g_\varepsilon|_{H^1(\Omega)}^2 + \varepsilon |\varphi(g) - \varphi_\varepsilon(g_\varepsilon)|_{L^2(\Omega)}^2 + |\varphi(g) - \varphi_\varepsilon(g_\varepsilon)|_{H^{-1}(\Omega)}^2 \leq C \varepsilon m(\Omega_{g,\varepsilon}) \varepsilon^{(1+p)/(1-p)}.
\]

(2.14a)

Furthermore, if \( \varphi(s) \leq \varphi_\varepsilon(s) \) for all \( s \in (0, \varepsilon^{1/(1-p)}) \), then

\[
g_\varepsilon \geq g \quad \text{in } \Omega,
\]

and hence if \( g, f \geq 0 \) then \( u_{k,\varepsilon}, v_{k,\varepsilon}, u_\varepsilon \geq 0 \) in \( Q_T \).

**Proof.** From Theorem 2.1 in [9] it follows that \( |g - g_\varepsilon|_{L^\infty(\Omega)} \leq \varepsilon^{1/(1-p)} \). Furthermore we have from (2.3a) that

\[
C |\varphi(g) - \varphi_\varepsilon(g_\varepsilon)|_{H^{-1}(\Omega)}^2 \leq |g - g_\varepsilon|_{H^1(\Omega)}^2 + C \varepsilon |\varphi(g) - \varphi_\varepsilon(g_\varepsilon)|_{L^2(\Omega)}^2
\]

\[
\leq |g - g_\varepsilon|_{H^1(\Omega)}^2 + \langle \varphi(g) - \varphi_\varepsilon(g_\varepsilon), \zeta - g \rangle + \langle \varphi(g) - \varphi_\varepsilon(g_\varepsilon), \zeta - g \rangle
\]

\[
\leq C \varepsilon^{-1} |\zeta - g|_{L^2(\Omega)}^2 \leq C m(\Omega_{g,\varepsilon}) \varepsilon^{(1+p)/(1-p)},
\]

where \( \zeta \equiv \varphi^{-1}_\varepsilon(\varphi(g)) \) if \( \varphi(g) \in (0, \varphi(\varepsilon^{1/(1-p)}) \) and \( \zeta \equiv g \) otherwise. By elliptic regularity \( |g - g_\varepsilon|_{H^2(\Omega)} \leq C |\varphi(g) - \varphi_\varepsilon(g_\varepsilon)|_{L^2(\Omega)}^2 \). Hence the desired result: (2.14a).
Equation (2.14b) follows from noting that
\[ |\nabla [g - g_\varepsilon]|_{L^2(\Omega)}^2 = \langle \nabla (g - g_\varepsilon), \nabla [g - g_\varepsilon] \rangle = \langle \varphi_\varepsilon(g_\varepsilon) - \varphi(g), [g - g_\varepsilon] \rangle \leq \langle \varphi(g_\varepsilon) - \varphi(g), [g - g_\varepsilon] \rangle \leq 0. \]

Finally (2.14b) and Theorems 2.1 and 2.2 yield that if \( g, f \geq 0 \) then \( u_{k,\varepsilon}, v_{k,\varepsilon}, u_\varepsilon \geq 0 \) in \( Q_T \).

Throughout the rest of this paper \( g_\varepsilon \) is as defined by (2.13). Note that we will recover the use of \( g_\varepsilon = g \) and its interpolant in the finite element approximation if \( g \) satisfies (2.11b); see Lemma 3.5 and Remark 3.1.

Problem (P) is strongly related to a degenerate problem which has been investigated intensively, the (generalized) porous medium equation
\[ \partial_t w - \Delta [\beta(w)] = f \quad \text{in} \; Q_T, \]
where \( \beta : \mathbb{R} \to \mathbb{R} \) is continuous and strictly increasing and without loss of generality \( \beta(0) = 0 \). The (classical) porous medium equation is given by \( \beta(w) = |w|^m \) for some \( m > 1 \). A change of variables yields (1.4a) with \( \phi = \beta^{-1} \). Obviously (P) is of the form (1.4a). On the other hand, (1.4a) can be written in the form of (P), if we assume that \( \phi \) satisfies (1.9) and, as in [11], for some \( \alpha > 0 \)
\[ \phi'(s) \geq \alpha \quad \forall \; s \in \mathbb{R}, \]
where we allow for \( \phi'(0) = \infty \); as we can substitute \( \partial_t u \) by \( \alpha \partial_t u \) in the definition of (P), which amounts to substituting \( \varphi \) by \( \varphi/\alpha \) and scaling \( t \) by \( 1/\alpha \). Actually, we can even cast problem (1.4a) in the form of (P) if we only assume for every \( m > 0 \)
\[ \phi'(s) \geq \alpha(m) > 0 \quad \forall \; s \in [-m, m]. \]
This condition is satisfied if, e.g., \( \beta \in C^1(\mathbb{R}) \) and \( \beta'(s) > 0 \) for \( s \neq 0 \). This can be seen as follows. As already noted, we can substitute \( \partial_t u \) by \( \alpha \partial_t u \) in the definition of (P), and (P) without affecting the developed theory. In particular the bounds \( y, \pi \) for the \( u \)-component are independent of \( \alpha \) and \( \varphi \). Choose \( m = \max\{u_{L^\infty(\Omega)}, \pi_{L^\infty(\Omega)}\} \) and \( \alpha = \alpha(m) \) according to (2.17). This \( \alpha \) we take in the definition of (P) and \( \varphi \equiv \varphi(\alpha) \) defined by
\[ \varphi(\alpha)(s) \equiv \begin{cases} \phi(-m) + \alpha m & s \leq -m, \\ \phi(s) - \alpha s & |s| \leq m, \\ \phi(m) - \alpha m & s \geq m. \end{cases} \]
Then \( \varphi(\alpha) \) satisfies (1.9) if \( \phi \) does so, and as the solution of (P) fulfills \( |u|_{L^\infty(Q_T)} \leq m \), we have that \( \alpha u + \varphi(\alpha)(u) = \phi(u) \); i.e., the solution of (1.4a) is the solution (P).

The existence result for (P) in Theorem 2.3 is not new. It is quite comparable with the basic results for the generalized porous medium equation (compare, e.g., [13]). What is of importance for the following is the precise information about its approximation by (P).

3. A continuous in time finite element approximation. We now consider the continuous piecewise linear finite element approximation to (P). Assuming (D3), we introduce
\[ S^h \equiv \{ \chi \in C(\Omega^h) : \chi|_\kappa \text{ is linear} \; \forall \; \kappa \in T^h \} \]
and

\[ S_0^h \equiv \{ \chi \in S^h : \chi = 0 \text{ on } \partial \Omega^h \}. \]

In the analysis that follows we extend \( \chi \in S^h \) from \( \Omega^h \) to \( \Omega \) \( \setminus \Omega^h \) by zero. Let \( \pi_h : C_0^0(\Omega) \to S^h \) denote the interpolation operator such that for any \( w \in C_0^0(\Omega) \), \( \pi_h w \in S^h \) satisfies

\[ (\pi_h w)(x_i) = w(x_i) \quad \forall \text{ nodes } x_i \text{ of the partition } T^h. \]

Let \( P_0^1 : L^2(\Omega) \to S^h \) denote the \( L^2 \) projection such that for any \( w \in L^2(\Omega) \), \( P_0^1 w \in S^h \) satisfies

\[ \langle w - P_0^1 w, \chi \rangle = 0 \quad \forall \chi \in S^h. \]

Let \( P_1^0 : H_0^1(\Omega) \to S_0^h \) denote the \( H^1 \) seminorm projection such that for any \( w \in H_0^1(\Omega) \), \( P_1^0 w \in S_0^h \) satisfies

\[ \langle \nabla (w - P_1^0 w), \nabla \chi \rangle = 0 \quad \forall \chi \in S_0^h. \]

We recall the standard approximation results for all \( \kappa \in T^h \)

\[ |w - \pi_h w|_{W^m,q(\kappa)} \leq Ch^{2-m} |w|_{W^m,q(\kappa)} \quad \text{for } m = 0 \text{ and } 1 \text{ and } q \in [1, \infty] \text{ if } d \leq 2 \text{ and } q \in (3/2, \infty) \text{ if } d = 3, \]

\[ (3.1a) \]

\[ |w - P_0^1 w|_{L^2(\Omega)} \leq Ch^m |w|_{H^m(\Omega)} \quad \text{for } m = 0, 1, \text{ and } 2, \]

\[ (3.1b) \]

\[ |w - P_1^0 w|_{L^2(\Omega)} + h|w - P_1^0 w|_{H^1(\Omega)} \leq Ch^m |w|_{H^m(\Omega)} \quad \text{for } m = 1 \text{ and } 2, \]

\[ (3.1c) \]

where in (3.1a) we note the imbedding \( W^{2,1}(\kappa) \subset C^0(\pi) \) in the case \( d=2 \); see, for example, page 300 in [8]. Another result that will be useful later is that

\[ |(I - \pi_h)\varphi(\chi)|_{L^2(\Omega)} \leq Ch |\nabla \pi_h[\varphi(\chi)]|_{L^2(\Omega)} \quad \forall \chi \in S_0^h; \]

\[ (3.2) \]

see [6, p. 68].

The standard Galerkin approximation to \((P_{k,\varepsilon})\) is then

\( (P_{k,\varepsilon}^h) \quad \text{Find } u_{k,\varepsilon}^h \in H^1(0, T; S_0^h) \text{ and } v_{k,\varepsilon}^h \in H^1(0, T; S^h) \text{ such that} \)

\[ \langle \partial_t u_{k,\varepsilon}^h + \partial_t v_{k,\varepsilon}^h, \chi \rangle + \langle \nabla u_{k,\varepsilon}^h, \nabla \chi \rangle = \langle f, \chi \rangle \quad \forall \chi \in S_0^h; \]

\[ \langle \partial_t v_{k,\varepsilon}^h, \chi \rangle = k \langle \varphi(\varepsilon u_{k,\varepsilon}^h) - v_{k,\varepsilon}^h, \chi \rangle \quad \forall \chi \in S^h, \]

\[ u_{k,\varepsilon}^h(\cdot, 0) = P_1^h g_\varepsilon(\cdot), \quad v_{k,\varepsilon}^h(\cdot, 0) = P_0^1[\varphi(\varepsilon g_\varepsilon(\cdot))]. \]

The above approximation is not practical since it requires the term \( \langle \varphi(\varepsilon u_{k,\varepsilon}^h), \chi \rangle \) to be integrated exactly. This is obviously difficult in practice, and it is computationally more convenient to consider a scheme where numerical integration is applied to all the terms and the initial data are interpolated as opposed to being projected. Below we introduce and analyze such a scheme.

For all \( w_1, w_2 \in C^0(\Omega^c) \) we set

\[ \langle w_1, w_2 \rangle^h \equiv \int_{\Omega^c} \pi_h(w_1 w_2) \]
as an approximation to \( \langle w_1, w_2 \rangle \). On setting
\[
|w|_h \equiv |(w, w)_h|^{1/2} \quad \text{for } w \in C^0(\Omega^h),
\]
we recall the well-known results
\[
|\chi|_{L^2(\Omega_h)} \leq |\chi|_h \leq C_1|\chi|_{L^2(\Omega_h)} \quad \forall \chi \in S^h
\]
and for \( m = 0 \) or \( 1 \)
\[
(3.3a) \quad \left| \int_{\Omega_h} \chi_1 \chi_2 - \langle \chi_1, \chi_2 \rangle^h \right| \leq C_2 h^{1+m} \| \chi_1 \|_{H^1(\Omega_h)} \| \chi_2 \|_{H^m(\Omega_h)} \quad \forall \chi_1, \chi_2 \in S^h.
\]

Assuming (D4), a more practical approximation to \((P_{k,\varepsilon})\) than \((P^h_{k,\varepsilon})\) is then
\[
\hat{\varepsilon}^h_{k,\varepsilon} \in H^1(0, T; S^h_0) \quad \text{and} \quad \hat{v}^h_{k,\varepsilon} \in H^1(0, T; S^b) 
\]
such that
\[
\begin{align*}
\langle \partial_t \hat{u}^h_{k,\varepsilon} + \partial_t \hat{v}^h_{k,\varepsilon}, \chi \rangle^h + \langle \nabla \hat{u}^h_{k,\varepsilon}, \nabla \chi \rangle^h &= (f, \chi)^h \quad \forall \chi \in S^h_0, \\
\langle \partial_t \hat{v}^h_{k,\varepsilon}, \chi \rangle^h &= k \langle \varphi(\hat{u}^h_{k,\varepsilon}) - \hat{v}^h_{k,\varepsilon}, \chi \rangle^h \quad \forall \chi \in S^h, \\
\hat{u}^h_{k,\varepsilon}(\cdot, 0) &= g^h_{\varepsilon}(\cdot), \quad \hat{v}^h_{k,\varepsilon}(\cdot, 0) = \pi_h[\varphi(\hat{u}^h_{\varepsilon}(\cdot))],
\end{align*}
\]
where \( g^h_{\varepsilon} \in S^h_0 \) is a suitable approximation to \( g_{\varepsilon} \), the unique solution of (2.13), satisfying \( |(\nabla g^h_{\varepsilon}, \nabla \chi)| \leq C|\chi|_{L^2(\Omega)} \) for all \( \chi \in S^h_0 \). Hence it follows that \( g^h_{\varepsilon} \in H^1(\Omega), |\varphi(g^h_{\varepsilon})| \leq C(h, k) \).

**Theorem 3.1.** Let assumption (D4) hold. Then for all \( \varepsilon \in (0, \varepsilon_0) \), \( h > 0 \) there exists a unique solution \( \{\hat{u}^h_{k,\varepsilon}, \hat{v}^h_{k,\varepsilon}\} \) to \((\hat{P}^h_{k,\varepsilon})\) and \( |\hat{u}^h_{k,\varepsilon}||_{L^\infty(Q_T)}, |\hat{v}^h_{k,\varepsilon}||_{L^\infty(Q_T)} \leq C(h, k) \).

**Proof.** The proof is provided in Theorem 3.1 in Part I [3].

**Lemma 3.1.** Let assumption (D4) hold. Then we have for all \( \varepsilon \in (0, \varepsilon_0) \), \( h > 0 \), and \( t \in (0, T) \) that
\[
|\nabla \hat{u}^h_{k,\varepsilon}(\cdot, t)|^2_{L^2(\Omega)} + |\partial_t \hat{u}^h_{k,\varepsilon}(\cdot, t)|^2_{L^2(Q_T)} + \varepsilon|\partial_t \hat{v}^h_{k,\varepsilon}(\cdot, t)|^2_{L^2(Q_T)} \leq C(k, \varepsilon, \varepsilon_0, 0)
\]
\[
+k^{-1} \left| \partial_t \hat{u}^h_{k,\varepsilon}(\cdot, t) + \pi_h[\varphi(\hat{u}^h_{k,\varepsilon}(\cdot))]|_{L^2(\Omega)} + |\nabla \partial_t \hat{u}^h_{k,\varepsilon}(\cdot, t)|_{L^2(Q_T)} \right| \leq C.
\]

**Proof.** Noting the bounds on \( g^h_{\varepsilon} \) above, see Lemma 3.1 in Part I [3].

In order to analyze the approximation \((\hat{P}^h_{k,\varepsilon})\) it is convenient to introduce an associated linear problem of \((\hat{P}^h_{k,\varepsilon})\):
\[
(P^h_{k,\varepsilon}) \quad \text{Find } u^{h,*}_{k,\varepsilon} \in H^1(0, T; S^h_0) \quad \text{and} \quad v^{h,*}_{k,\varepsilon} \in H^1(0, T; S^b) 
\]
such that
\[
\begin{align*}
\langle \partial_t u^{h,*}_{k,\varepsilon} + \partial_t v^{h,*}_{k,\varepsilon}, \chi \rangle + \langle \nabla u^{h,*}_{k,\varepsilon}, \nabla \chi \rangle &= (f, \chi) \quad \forall \chi \in S^h_0, \\
\langle \partial_t v^{h,*}_{k,\varepsilon}, \chi \rangle &= k \langle \varphi(u^{h,*}_{k,\varepsilon}) - v^{h,*}_{k,\varepsilon}, \chi \rangle \quad \forall \chi \in S^h, \\
u^{h,*}_{k,\varepsilon}(\cdot, 0) &= P^h_1 g_{\varepsilon}(\cdot), \quad v^{h,*}_{k,\varepsilon}(\cdot, 0) = P^h_0[\varphi(\hat{u}^h_{\varepsilon}(\cdot))].
\end{align*}
\]

The existence and uniqueness of \( \{u^{h,*}_{k,\varepsilon}, v^{h,*}_{k,\varepsilon}\} \) solving \((P^h_{k,\varepsilon})\) for all \( \varepsilon \in (0, \varepsilon_0) \) and \( h > 0 \) is easily established under assumption (D3), and we have the following result.
LEMMA 3.2. Let assumption (D3) hold. Then we have for all \( \varepsilon \in (0, \varepsilon_0] \), \( h > 0 \), and \( t \in (0, T) \) that

\[
\left| u_{k,\varepsilon} - u_{k,\varepsilon}^h \right|_{L^2(Q_t)}^2 + h^2 \left| \nabla \int_0^t (u_{k,\varepsilon} - u_{k,\varepsilon}^h)(\cdot, s) \, ds \right|_{L^2(\Omega)}^2 
\leq C h^4 \left[ \left| u_{k,\varepsilon} \right|_{L^2(0,t;H^2(\Omega))}^2 + \left| g_{\varepsilon} \right|_{H^2(\Omega)}^2 \right] \leq C \varepsilon^{-1} h^4.
\]

Proof. The first inequality in (3.5) is proved in Lemma 3.2 of Part I [3]. Under the stated assumptions on \( \Omega \) we have that

\[
\left| u_{k,\varepsilon} \right|_{L^2(0,T;H^2(\Omega))} \leq C \left[ \left| \partial_t u_{k,\varepsilon} \right|_{L^2(Q_T)} + \left| \partial_x u_{k,\varepsilon} \right|_{L^2(Q_T)} + \left| f \right|_{L^2(Q_T)} \right] \leq C \varepsilon^{-1/2},
\]

where we have noted (2.7). Hence the desired result (3.5).

Assuming (D5), it is easy to deduce that the stiffness matrix \( \left\{ (\nabla X_i, \nabla X_j) \right\}_{i,j=1}^{3} \), where \( \{x_i\}_{i=1}^{3} \) are the internal nodes of the partition and \( X_j \in S_0^h \) is such that \( X_j(x_i) = \delta_{ij}, \ 1 \leq i,j \leq 3 \), is an M-matrix. From this property one can deduce that

\[
M^{-1} \left| \nabla \pi_h(\varphi(\chi)) \right|_{L^2(\Omega)}^2 \leq \langle \nabla \chi, \nabla \pi_h(\varphi(\chi)) \rangle \quad \forall \chi \in S_0^h;
\]

see, e.g., section 2.4.2 of [10]. Furthermore it follows from (3.7), (3.1a), (2.3a), and (3.2) that, for all \( w \in H^2(\Omega) \cap H_0^1(\Omega), \)

\[
M^{-1} \left| \nabla \pi_h(\varphi_w(\varepsilon)) \right|_{L^2(\Omega)}^2 \leq \langle \nabla w, \nabla \varphi_w(\varepsilon) \rangle + C \varepsilon^{-1} h^2 \left| w \right|_{H^2(\Omega)}^2;
\]

see the derivation of (3.12) in Part I [3] for details.

COROLLARY 3.1. Let assumption (D5) hold. Then the unique solution \( \{u_{k,\varepsilon}^h, \varphi_{k,\varepsilon}^h\} \) to \( \left( \hat{P}_{k,\varepsilon}^h \right) \), \( \varepsilon \in (0, \varepsilon_0] \), and \( h > 0 \) satisfies the bounds (2.5) with \( u, \pi, v, \) and \( \bar{v} \in C(\Omega) \) all independent of \( \varepsilon, k, \) and \( h \). In particular, if \( g_\varepsilon \) and \( f \geq 0 \) then \( \hat{u}_{k,\varepsilon}^h, \hat{v}_{k,\varepsilon}^h \geq 0 \) in \( Q_T \).

Proof. See Corollary 3.1 in Part I [3] and compare the proof of Theorem 2.1 for a justification of the time independence of the bounds.

LEMMA 3.3. Under assumption (D5) we have for all \( \varepsilon \in (0, \varepsilon_0] \) and \( h > 0 \), provided \( \varepsilon^{-1} kh^2 \leq C \), and for all \( t \in (0, T) \) that

\[
\left| u_{k,\varepsilon}^h - \hat{u}_{k,\varepsilon}^h \right|_{E^2(k,t)}^2 + \varepsilon \left| \pi_h(\varphi_{\varepsilon}(u_{k,\varepsilon}) - \varphi_{\varepsilon}(\hat{u}_{k,\varepsilon})) \right|_{L^2(Q_t)}^2 
\leq C \varepsilon^{-1} + \left| g_\varepsilon^h \right|_{H^1(\Omega)}^2 h^2 + C \left| P^1_k g_{\varepsilon} - g_\varepsilon^h \right|_{L^2(\Omega)}^2 
+ \sup_{\chi \in S_0^h} \{ \left| \left| \varphi_{\varepsilon}(g_{\varepsilon}), \chi \right| - \left| \varphi_{\varepsilon}(g_{\varepsilon}^h), \chi \right| \right|_{H^1(\Omega)}^2 \}.
\]

Proof. The proof is provided in Lemma 3.3 of Part I [3].

We now improve on the bound (3.9) in the physically interesting case of given data \( g \) and \( f \geq 0 \) yielding \( u \geq 0 \) in \( Q_T \). Assuming (D6), we set \( \varphi_{\varepsilon} \) to be the following quadratic regularization of \( \varphi \):

\[
\varphi_{\varepsilon}(s) \equiv \begin{cases} \alpha s^2 + \beta s & \text{for } s \in [0, \delta], \\ \varphi(s) & \text{otherwise}, \end{cases}
\]

where \( \alpha \equiv \delta^{-1} \varphi'(\delta) - \delta^{-2} \varphi(\delta) \), \( \beta \equiv -\varphi'(\delta) + 2\delta^{-1} \varphi(\delta) \), and \( \delta \equiv \varepsilon^{1/(1-p)} \) so that \( \varphi_{\varepsilon} \in C^1[0, \infty) \). From (D6)(iii) it follows that \( \varphi(\delta) \geq \delta \varphi'(\delta) \), which in turn yields
that $0 < b \leq C_1 \varepsilon^{-1}$ and $-C_2 \varepsilon^{(p-2)/(1-p)} \leq a \leq 0$, (see (2.3b)), and hence $\varphi_\varepsilon$ satisfies the condition (2.2). Therefore all the results proved so far in this paper hold under assumption (D6). We note for example that $\varphi(s) \equiv \alpha s^p$ for $s \geq 0$ with $p \in (0, 1)$ and $\alpha \in \mathbb{R}^+$ satisfies (1.9) and (D6)(iii).

Assuming (D6), it follows for all $w \in H^1_0(\Omega) \cap W^{2,1}(\Omega)$ with $w(x) \in [0, m]$ for $x \in \Omega$ that

$$|\varphi_\varepsilon(w)|_{W^{2,1}(\Omega)} \leq C \varepsilon^{-1} \|w\|_{W^{2,1}(\Omega)};$$

see (3.25) in Part I [3]. From (D6)(i) we have the discrete Sobolev imbedding

$$|\chi|_{L^\infty(\Omega)} \leq C[\ln(1/h)]^r \|\chi\|_{H^r(\Omega)} \leq C[\ln(1/h)]^r \|\nabla \chi\|_{L^2(\Omega)} \quad \forall \chi \in S^h,$$

where $r = 0$ if $d = 1$ and $r = 1/2$ if $d = 2$; see, for example, page 67 in [14]. As noted in Part I [3], the quasi-uniformity restriction is not really restrictive in practice.

**Lemma 3.4.** Under assumption (D6) we have for all $\varepsilon \in (0, \varepsilon_0]$ and $h > 0$, provided $\varepsilon^{-1}kh^2 \leq C$, and for all $t \in (0, T]$ that

$$\|u_{k,\varepsilon}^h - \hat{u}_{k,\varepsilon}^h\|_{E_2(\Omega)}^2 + C\|\varphi_\varepsilon(u_{k,\varepsilon}) - \varphi_\varepsilon(\hat{u}_{k,\varepsilon})\|_{E_2(Q_t)}^2 \\
\leq Ckh^2[\varepsilon^{-2}[\ln(1/h)]^{2r} ||u_{k,\varepsilon}||_{L^2(0,T;W^{2,1}(\Omega))}^2 + ||g_\varepsilon^h||_{H^1(\Omega)}^2]$$

$$+ C\|P_h^1 g_\varepsilon - g_\varepsilon^h\|_{L^2(\Omega)}^2 + \sup_{\chi \in S^h} \{[\langle \varphi_\varepsilon(g_\varepsilon), \chi \rangle - \langle \varphi_\varepsilon(g_\varepsilon^h), \chi \rangle \|^h_{H^1(\Omega)}] \}^2. (3.13)$$

**Proof.** The proof is provided in Lemma 3.4 of Part I [3].

One can also consider the corresponding approximation without relaxing the reaction:

**Theorem 3.2.** Let assumption (D4) hold. Then for all $\varepsilon \in (0, \varepsilon_0]$ and $h > 0$ there exists a unique solution $\hat{u}_{\varepsilon}^h$ to $(\hat{P}_{\varepsilon}^h)$. Moreover, for all $t \in (0, T]$ we have that

$$\|\hat{u}_\varepsilon^h - \hat{u}_{k,\varepsilon}^h\|_{E_2(\Omega)}^2 + C\|\varphi_\varepsilon(\hat{u}_\varepsilon^h) - \varphi_\varepsilon(\hat{u}_{k,\varepsilon}^h)\|_{E_2(Q_t)}^2 \\
\leq Ck^{-2} ||\partial_t \hat{u}_{k,\varepsilon}^h||_{L^2(\Omega_T)}^2 \leq C\varepsilon^{-1}k^{-2}. (3.14)$$

In addition, under assumption (D5) $\hat{u}_{\varepsilon}^h$ satisfies the bounds in (2.8a). In particular, if $g_\varepsilon^h$ and $f \geq 0$ then $\hat{u}_\varepsilon^h \geq 0$ in $Q_T$.

**Proof.** Existence and uniqueness of a solution to $(\hat{P}_{\varepsilon}^h)$ follows from a discrete analogue of Theorem 2.2. The first inequality in (3.14) is a discrete analogue of the first inequality in (2.8b). The second inequality in (3.14) follows from (3.4). The bounds in (2.8a) follow from (3.14), the equivalence of norms on $S^h$, and Corollary 3.1.

**Lemma 3.5.** Let assumption (D4) hold. Then for all $\varepsilon \in (0, \varepsilon_0]$ and $h > 0$ let $\varepsilon \in (0, \varepsilon]$ and $\hat{h} \in (0, h]$ be such that

$$m(\Omega_{\varepsilon,\hat{h}}) \|\varepsilon\|_{E_{2p/(1-p)}} \leq C_E(T)\varepsilon^{(1+p)/(1-p)}, (3.15a)$$

$$\hat{h} \leq C(\hat{\varepsilon}/\varepsilon)^{1/2}h, (3.15b)$$
where $\mu = 1$ if $d = 1$, $\mu > 1$ if $d = 2$, and $\mu = 6/5$ if $d = 3$. Let $T^h$ be a subpartitioning of $\mathcal{T}^h$ consisting of regular simplices $\bar{k}$ with $h_{\bar{k}} \equiv \text{diam}(\bar{k})$ and $\hat{h} \equiv \max_{\bar{k} \in \mathcal{T}^h} h_{\bar{k}}$, such that $\Omega^h \equiv \cup_{\bar{k} \in \mathcal{T}^h} \Omega_{\bar{k}} \equiv \cup_{\bar{k} \in \mathcal{T}^h} \bar{k}$ and $\bar{k} = \cup_{\bar{k}}$ for all $k \in T^h$. Let $S^h$ be the associated continuous piecewise linear space, $\pi^h : C^0(\Omega^h) \to S^h$ be the corresponding interpolation operator, and $\hat{\Omega}_{g,e} \equiv \cup_{\bar{k} \in \mathcal{T}^h} \bar{k}$, where $\mathcal{T}^h_{g,e} \equiv \{ k \in \mathcal{T}^h : k \cap \Omega_{g,e} \neq \emptyset \}$.

Then we have on choosing $g^h_{\varepsilon} \in S^h_{0}$ to be the unique solution of

$$
\langle \nabla g^h_{\varepsilon}, \nabla \chi \rangle + \langle \varphi_{\varepsilon}(g^h_{\varepsilon}), \chi \rangle^h
$$

(3.16a)

$$
= \langle \nabla \pi^h g, \nabla \chi \rangle + \langle \varphi_{\varepsilon}(g), \chi \rangle^h + \int_{\hat{\Omega}_{g,e}} \pi^h [\varphi_{\varepsilon}(g) - \varphi_{\varepsilon}(g)] \chi \quad \forall \chi \in S^h_{0}
$$

that $|\langle \nabla g^h_{\varepsilon}, \nabla \chi \rangle| \leq C|\chi|_{L^2(\Omega)}$ for all $\chi \in S^h_{0}$ as required in $(\hat{P}_{k,e})$. In addition if assumption (D5) holds and $\varphi_{\varepsilon}(s) \geq \varphi_{\varepsilon}(s)$ for all $s \in (0,\varepsilon^{1/(1-p)})$, then

$$
g^h_{\varepsilon} \geq \pi^h g \quad \text{in } \Omega,
$$

and hence if $g, \: f \geq 0$ then $u^h_{k,e}, \: \hat{v}^h_{k,e}, \: \hat{u}^h_{k,e} \geq 0$ in $Q_T$. Furthermore, provided $\varepsilon^{-1}h^2 \leq C$, we have that

$$
|P^1_k g^h_{\varepsilon} - g^h_{\varepsilon}^h|^2_{L^2(\Omega)} + \sup_{\chi \in S^h_{0}} \left\{ |\langle \varphi_{\varepsilon}(g^h_{\varepsilon}), \chi \rangle - \langle \varphi_{\varepsilon}(g^h_{\varepsilon}), \chi \rangle^h | \right\}/ \| \chi \|_{H^1(\Omega)}^2 \leq
$$

$$
\leq C A_{\varepsilon}(T)^{2(1+p)/(1-p)} + \left\{ \begin{array}{ll}
C_{\varepsilon}^{-1}h^2 & \text{if (D5)} \\
C_{\varepsilon}^{-2}h^4[\ln(1/h)]^{2r} & \text{if (D6)}
\end{array} \right.
$$

(3.17)

Proof. Choosing $\chi \equiv g^h_{\varepsilon}$ in (3.16a) yields that $\| g^h_{\varepsilon} \|_{H^1(\Omega)} \leq C$, as $g \in H^2(\Omega)$, and hence $|\varphi_{\varepsilon}(g^h_{\varepsilon})| \leq C$. By noting that $|\langle \nabla \pi^h g, \nabla \chi \rangle| \leq |g|_{H^2(\Omega)}|\chi|_{L^2(\Omega)}$, (see, e.g., (3.4c) in Part I [3]), it follows that $|\langle \nabla g^h_{\varepsilon}, \nabla \chi \rangle| \leq C|\chi|_{L^2(\Omega)}$ for all $\chi \in S^h_{0}$. The result (3.16b) is a discrete analogue of (2.14b) and follows from choosing $\chi \equiv \sum_{j=1}^{l} [g(x_j) - g^h_{\varepsilon}(x_j)] \chi_j$ in (3.16a) and noting that the stiffness matrix is an $M$-matrix. Equation (3.16b) and Corollary 3.1 yield that if $g, \: f \geq 0$ then $u^h_{k,e}, \: \hat{v}^h_{k,e}, \: \hat{u}^h_{k,e} \geq 0$ in $Q_T$.

(From (2.13) and (3.16a) we have that

$$
\langle \nabla (P^1_k g - g^h_{\varepsilon} - \pi^h g), \nabla \chi \rangle + \left[ \langle \varphi_{\varepsilon}(g^h_{\varepsilon}), \chi \rangle - \langle \varphi_{\varepsilon}(g), \chi \rangle^h \right]
$$

(3.18a)

$$
= \langle \nabla (g - \varphi_{\varepsilon}(g)), \nabla \chi \rangle + \left[ \langle \varphi_{\varepsilon}(g), \chi \rangle - \langle \varphi_{\varepsilon}(g), \chi \rangle^h \right] + \int_{\hat{\Omega}_{g,e}} (I - \pi^h_k)[\varphi_{\varepsilon}(g) - \varphi_{\varepsilon}(g)] \chi \quad \forall \chi \in S^h_{0};
$$

that is,

$$
\langle \nabla (P^1_k d_{\varepsilon} - d^h_{\varepsilon}), \nabla \chi \rangle + \left[ \langle \varphi_{\varepsilon}(g), \chi \rangle - \langle \varphi_{\varepsilon}(g), \chi \rangle^h \right] = \langle \nabla (g - \varphi_{\varepsilon}(g)), \nabla \chi \rangle + \left[ \langle \varphi_{\varepsilon}(g), \chi \rangle - \langle \varphi_{\varepsilon}(g), \chi \rangle^h \right] + \int_{\hat{\Omega}_{g,e}} (I - \pi^h_k)[\varphi_{\varepsilon}(g) - \varphi_{\varepsilon}(g)] \chi \quad \forall \chi \in S^h_{0},
$$

(3.18b)

where $d_{\varepsilon} \equiv g - g \in H^2(\Omega)$, noting Corollary 2.1, and $d^h_{\varepsilon} \equiv g^h_{\varepsilon} - \pi^h g \in S^h_{0}$.}
Similarly to (2.11a) it follows that
\[
|\langle \varphi(g) - \varphi_\varepsilon(g), \chi \rangle| \leq C|\varphi(g) - \varphi_\varepsilon(g)|_{H^{-1}(\Omega)}\|\chi\|_{H^1(\Omega)}
\]
(3.19)
\[
\leq C(m_{\Omega,\varepsilon})^{1/\mu}\varepsilon^{\mu/(1-\mu)}\|\chi\|_{H^1(\Omega)}.
\]

Assuming (D5) it follows from (3.2), (3.1a), and (2.2c) that for all \(w \in H^2(\Omega) \cap H^1_0(\Omega)\) and for \(\varepsilon^{-1}h^2 \leq C\)
\[
|(I - \pi_h)\varphi_\varepsilon(w)|_{L^2(\Omega)} \leq |(I - \pi_h)\varphi_\varepsilon(\pi_h w)|_{L^2(\Omega)} + |\varphi_\varepsilon(w) - \varphi_\varepsilon(\pi_h w)|_{L^2(\Omega)}
\]
\[
\leq C h|\nabla|\pi_h\varphi_\varepsilon(w)|_{L^2(\Omega)} + C\varepsilon^{-1}h^2|w|_{H^2(\Omega)}
\]
(3.20a)
\[
\leq C\varepsilon^{-1/2}h[1 + \|w\|_{H^2(\Omega)}],
\]
since (3.8) implies that
\[
\varepsilon|\nabla|\pi_h\varphi_\varepsilon(w)|_{L^2(\Omega)}^2 \leq -C(\Delta w, \varphi_\varepsilon(w)) + C\varepsilon^{-1}h^2|w|^2_{H^2(\Omega)}
\]
(3.20b)
\[
\leq C[1 + \|w\|_{H^2(\Omega)}]^2.
\]
Hence (3.20a), (3.20b), and (3.3b) yield that
\[
|\langle \varphi_\varepsilon(w), \chi \rangle| \leq |\langle (I - \pi_h)\varphi_\varepsilon(w), \chi \rangle| + |\langle \pi_h\varphi_\varepsilon(w), \chi \rangle - \langle \varphi_\varepsilon(w), \chi \rangle| \leq C\varepsilon^{-1/2}h[1 + \|w\|_{H^2(\Omega)}]\|\chi\|_{H^1(\Omega)}.
\]
(3.21)

As (D5) holds also for \(T^h\) and (3.15) yields that \(\varepsilon^{-1}h^2 \leq \varepsilon^{-1}h^2 \leq C\varepsilon^{-1}h^2\), we have similarly to (3.20) that
\[
\int_{\Omega_h,\varepsilon} (I - \pi_h)[\varphi_\varepsilon(g) - \varphi_\varepsilon(g)]\chi \leq C\varepsilon^{-1/2}h[1 + \|g\|_{H^2(\Omega)}]\|\chi\|_{L^2(\Omega)}.
\]
(3.22)
Choosing \(\chi \equiv P_h^1 d_\varepsilon - d_\varepsilon^h\) in (3.18b) and noting (2.3a), (3.19), (3.21), (3.22), (3.15a), (3.15b), and (3.1c) yield that
\[
|P_h^1 d_\varepsilon - d_\varepsilon^h|_{H^1(\Omega)}^2 + C\varepsilon|\varphi_\varepsilon(g_\varepsilon) - \varphi_\varepsilon(g_\varepsilon^h)|_h^2
\]
\[
\leq |P_h^1 d_\varepsilon - d_\varepsilon^h|_{H^1(\Omega)}^2 + \langle \varphi_\varepsilon(g_\varepsilon) - \varphi_\varepsilon(g_\varepsilon^h), g_\varepsilon - g_\varepsilon^h\rangle_h
\]
\[
\leq \langle \varphi_\varepsilon(g_\varepsilon) - \varphi_\varepsilon(g_\varepsilon^h), (I - P_h^1) d_\varepsilon\rangle_h + CA_\varepsilon(T)\varepsilon^{(1+p)/(1-p)} + C\varepsilon^{-1}h^2
\]
(3.23)
\[
\leq CA_\varepsilon(T)\varepsilon^{(1+p)/(1-p)} + C\varepsilon^{-1}h^2.
\]

Under assumption (D6) we have in place of (3.20a), after noting (3.1a) and (3.11), that
\[
|(I - \pi_h)\varphi_\varepsilon(w)|_{L^1(\Omega)} \leq C h^2|\varphi_\varepsilon(w)|_{W^{2,1}(\Omega)} \leq C\varepsilon^{-1}h^2\|w\|_{W^{2,1}(\Omega)}.
\]
(3.24)
Hence in place of (3.21), we have on noting (3.12), (3.3b), and (3.20b) that
\[
|\langle \varphi_\varepsilon(w), \chi \rangle| \leq |\langle (I - \pi_h)\varphi_\varepsilon(w), \chi \rangle| + |\langle \pi_h\varphi_\varepsilon(w), \chi \rangle - \langle \varphi_\varepsilon(w), \chi \rangle| \leq C\varepsilon^{-1}h^2[\ln(1/h)]^{r}\|w\|_{H^2(\Omega)}\|\chi\|_{H^1(\Omega)}.
\]
(3.25)
Similarly, in place of (3.22) we have that
\[
\int_{\Omega_h,\varepsilon} |(I - \pi_h)[\varphi_\varepsilon(g) - \varphi_\varepsilon(g)]\chi| \leq C\varepsilon^{-1}h^2[\ln(1/h)]^{r}\|g\|_{H^2(\Omega)}\|\chi\|_{H^1(\Omega)}.
\]
(3.26)
Therefore, similarly to (3.23), it follows that
\[ |P^\dagger h d - d^h |_{H^1(\Omega)} + C\varepsilon |\varphi_\varepsilon (g) - \varphi_\varepsilon (g^h)|^2_h \]
\[ \leq C A_\varepsilon (T) \varepsilon^{1 + p} / (1 - p) + C \varepsilon^{-2} h^4 |\ln (1/h)|^{2r}. \]
(3.27)

Combining (3.18a), (3.19), (3.15), (3.21), (3.22), (3.23), (3.25), (3.26), and (3.27) yields the desired result (3.17).

**Remark 3.1.** First we note the construction of \( g^h \) as in (3.16a) is fully practical. Furthermore, in line with (2.11b) if \([m(\Omega, \gamma)]^{2/\mu} \leq C A_\varepsilon (T) \varepsilon \), then one can choose \( \varepsilon = \varepsilon \) and hence \( g^h \equiv \pi_h g \) in this case.

Second we note that (a) the linear regularization (2.4) and (b) the quadratic regularization (3.10) are such that
\[ \varepsilon \in (0, \varepsilon_0) \text{ and } h > 0, \text{ provided } \varepsilon^{-1} k h^2 \leq C, \text{ and } \forall t \in (0, T) \]
(i) under assumption (D5)
\[ \| u - \hat{u}^h_{k, \varepsilon} \|_{L^2(Q_t)}^2 \leq E \]
\[ \| u - \hat{u}^h_{k, \varepsilon} \|_{L^2(Q_t)}^2 \leq E, \quad \frac{1}{2} \left( \int_0^t (u - \hat{u}^h_{k, \varepsilon})(\cdot, s) ds \right)^2_{H^1(\Omega)} \leq \min \{ E, \varepsilon^{-1} h^2 \}, \]
(3.29a)
and
\[ \varepsilon |\varphi(u) - \varphi_\varepsilon (\hat{u}^h_{k, \varepsilon})|_{L^2(Q_t)}^2 \leq \min \{ E, h^2 \}, \]
(3.29b)
where
\[ \varepsilon \leq C (1 + \varepsilon^{1 + p} / (1 - p) \{ A_\varepsilon (T) + m(\Omega, \gamma) \} + \varepsilon^{-1} k h^2 \leq C \varepsilon^{-3} h^4 |\ln (1/h)|^{2r}, \]
\( r = 0 \text{ if } d = 1, \text{ and } r = 1/2 \text{ if } d = 2. \)

**Proof.** It follows similarly to (3.20) that
\[ |\varphi_\varepsilon (u_{k, \varepsilon}) - \varphi_\varepsilon (\hat{u}^h_{k, \varepsilon})|_{L^2(Q_t)}^2 \leq C h^2 |\nabla (\pi_h \varphi_\varepsilon (u_{k, \varepsilon}))|_{L^2(Q_t)}^2 + C \varepsilon^{-2} h^4 |u_{k, \varepsilon}|_{L^2 (0, t; H^2(\Omega))}, \]
(3.30a)
since (3.6), (2.6), (2.7), and (3.8) yield that
\[ \| u_{k, \varepsilon} \|_{L^2 (0, t; H^2(\Omega))} \leq C k \quad \text{and} \quad |\nabla (\pi_h \varphi_\varepsilon (u_{k, \varepsilon}))|_{L^2(Q_t)}^2 \leq C \varepsilon^{-1}. \]
(3.30b)
The results (3.28) and (3.29) then follow immediately from combining (2.10), (2.14a), (2.8b), (3.5), (3.6), (3.17), (3.30a) with (3.9) and (3.13), respectively.

**Corollary 3.3a.** Let assumption (D5) hold. Then for all \( h > 0 \) and \( t \in (0, T) \]
(i) under no assumptions on nondegeneracy and on choosing \( \varepsilon = Ch^{1 - p} \leq \varepsilon_0 \) and \( k = Ch^{-1} \), we have that
\[ |u - \hat{u}^h_{k, \varepsilon}|_{L^2(Q_T)} + \frac{1}{2} \left( \int_0^t (u - \hat{u}^h_{k, \varepsilon})(\cdot, s) ds \right)^2_{H^1(\Omega)} \leq Ch^{(1 + p)/2}, \]
(3.31a)
and
\begin{equation}
(3.31b) \quad |\varphi(u) - \pi_h[\varphi_z(\tilde{u}_{k,e}^h)]|_{L^2(Q_T)} \leq C h^p.
\end{equation}

(ii) Assuming (N.D.) and that \( m(\Omega_{q,\varepsilon}) \leq C \varepsilon \), then on choosing \( \varepsilon = C h^{2(1-p)/(3-p)} \leq \varepsilon_0 \) and \( k = C h^{-1} \) we have that
\begin{equation}
(3.32a) \quad |u - \tilde{u}_{k,e}^h|_{L^2(Q_T)} + \left\| \int_0^t (u - \tilde{u}_{k,e}^h)(\cdot,s) \, ds \right\|_{H^1(\Omega)} \leq C h^{2/(3-p)}
\end{equation}
and
\begin{equation}
(3.32b) \quad |\varphi(u) - \pi_h[\varphi_z(\tilde{u}_{k,e}^h)]|_{L^2(Q_T)} \leq C h^{(1+p)/(3-p)}.
\end{equation}

**Proof.** The results follow directly from (3.28). \( \square \)

**COROLLARY 3.3b.** Let assumption (D6) hold. Then for all \( h > 0 \) and \( t \in (0,T] \)
(i) under no assumptions on nondegeneracy and on choosing
\[ \varepsilon = C\{h^2[\ln(1/h)]^r\}^{2(1-p)/(5-2p)} \leq \varepsilon_0 \]
and
\( k = C\{h^2[\ln(1/h)]^r\}^{-2/(5-2p)} \)
we have for all \( p \in (1/2,1] \) that
\begin{equation}
(3.33a) \quad |u - \tilde{u}_{k,e}^h|_{L^2(Q_T)} \leq C\{h^2[\ln(1/h)]^r\}^{(1+p)/(5-2p)},
\end{equation}
\begin{equation}
(3.33b) \quad \left\| \int_0^t (u - \tilde{u}_{k,e}^h)(\cdot,s) \, ds \right\|_{H^1(\Omega)} \leq C\{h^2[\ln(1/h)]^r\}^{3/[2(5-2p)]},
\end{equation}
and
\begin{equation}
(3.33c) \quad |\varphi(u) - \pi_h[\varphi_z(\tilde{u}_{k,e}^h)]|_{L^2(Q_T)} \leq C\{h^2[\ln(1/h)]^r\}^{q/(5-2p)},
\end{equation}
where \( q = \min\{2p,3/2\} \).

(ii) Assuming (N.D.) and that \( m(\Omega_{q,\varepsilon}) \leq C \varepsilon \), then on choosing
\[ \varepsilon = C\{h^2[\ln(1/h)]^r\}^{4(1-p)/(13-7p)} \leq \varepsilon_0 \]
and
\[ k = C\{h^2[\ln(1/h)]^r\}^{-2/(13-7p)} \]
we have for all \( p \in (1/3,1] \) that
\begin{equation}
(3.34a) \quad |u - \tilde{u}_{k,e}^h|_{L^2(Q_T)} \leq C\{h^2[\ln(1/h)]^r\}^{4/(13-7p)},
\end{equation}
\begin{equation}
(3.34b) \quad \left\| \int_0^t (u - \tilde{u}_{k,e}^h)(\cdot,s) \, ds \right\|_{H^1(\Omega)} \leq C\{h^2[\ln(1/h)]^r\}^{3(3-p)/(2(13-7p))},
\end{equation}
and
\begin{equation}
(3.34c) \quad |\varphi(u) - \pi_h[\varphi_z(\tilde{u}_{k,e}^h)]|_{L^2(Q_T)} \leq C\{h^2[\ln(1/h)]^r\}^{q/(13-7p)},
\end{equation}
where \( q = \min\{2(p+1),3(3-p)/2\} \).

**Proof.** The results follow directly from (3.29). \( \square \)
there exists a unique solution

\[ \hat{u}_{k,\varepsilon} \text{ and } \hat{\tau} \text{ time step} \]

then (\( \hat{P}^{h}_{k,\varepsilon} \)) can be restated: for almost every \( t \in (0, T] \),

\[ \langle \partial_t \hat{U}^{L}_{k,\varepsilon}(\cdot, 0), \hat{\chi} \rangle = g_{h}^{\varepsilon}(\cdot) \quad \text{and} \quad \hat{V}^{L}_{k,\varepsilon}(\cdot, 0) = \pi_{h}[\varphi_{\varepsilon}(g_{h}^{\varepsilon}(\cdot))]. \]

where \( f^{n}(\cdot) \equiv f(\cdot, n\tau) \).

4. A fully discrete and practical finite element approximation. In this section we analyze the following fully discrete practical approximation to (\( \hat{P}_{k,\varepsilon} \)) with time step \( \tau = T/N \):

\[ (\hat{P}^{h}_{k,\varepsilon}) \text{ For } n = 1 \to N \text{ find } \hat{u}_{k,\varepsilon}^{h,n} \in S_{0}^{h} \text{ and } \hat{v}_{k,\varepsilon}^{h,n} \in S^{h} \text{ such that} \]

\[ \tau^{-1} \left( \langle \hat{u}_{k,\varepsilon}^{h,n} - \hat{u}_{k,\varepsilon}^{h,n-1}, \hat{\chi} \rangle + \langle \hat{v}_{k,\varepsilon}^{h,n} - \hat{v}_{k,\varepsilon}^{h,n-1}, \hat{\chi} \rangle \right) = \langle f^{n}, \chi \rangle^{h} \quad \forall \chi \in S_{0}^{h}, \]

\[ \tau^{-1} \langle \hat{u}_{k,\varepsilon}^{h,0}, \hat{\chi} \rangle = k \langle \varphi_{\varepsilon}(\hat{u}_{k,\varepsilon}^{h,n}) - \hat{v}_{k,\varepsilon}^{h,n}, \chi \rangle^{h} \quad \forall \chi \in S^{h}, \]

\[ \hat{u}_{k,\varepsilon}^{h,0}(\cdot) = g_{h}^{\varepsilon}(\cdot) \quad \hat{v}_{k,\varepsilon}^{h,0}(\cdot) = \pi_{h}[\varphi_{\varepsilon}(g_{h}^{\varepsilon}(\cdot))]. \]

Then (\( \hat{F}^{h,\tau}_{k,\varepsilon} \)) can be restated: for almost every \( t \in (0, T] \),

\[ \langle \partial_t \hat{U}^{L}_{k,\varepsilon} + \partial_t \hat{V}^{L}_{k,\varepsilon}, \hat{\chi} \rangle^{h} + \langle \nabla \hat{U}^{L}_{k,\varepsilon}, \nabla \hat{\chi} \rangle^{h} = \langle \hat{f}, \hat{\chi} \rangle^{h} \quad \forall \chi \in S_{0}^{h}, \]

\[ \langle \partial_t \hat{V}^{L}_{k,\varepsilon}, \hat{\chi} \rangle^{h} = k \langle \varphi_{\varepsilon}(\hat{u}_{k,\varepsilon}^{h,n}) - \hat{V}^{L}_{k,\varepsilon}, \hat{\chi} \rangle^{h} \quad \forall \chi \in S^{h}, \]

\[ \hat{U}^{L}_{k,\varepsilon}(\cdot, 0) = g_{h}^{\varepsilon}(\cdot) \quad \hat{V}^{L}_{k,\varepsilon}(\cdot, 0) = \pi_{h}[\varphi_{\varepsilon}(g_{h}^{\varepsilon}(\cdot))]. \]

where \( \hat{f}(\cdot, t) \equiv f^{n}(\cdot) \equiv f(\cdot, n\tau) \) if \( t \in ([n - 1]\tau, n\tau] \), \( n = 1 \to N \).

Theorem 4.1. Let assumption (D4) hold. Then for all \( \varepsilon \in (0, \varepsilon_{0}] \), \( h \) and \( \tau > 0 \), there exists a unique solution \( \{\hat{U}_{k,\varepsilon}, \hat{V}_{k,\varepsilon}\} \) to (\( \hat{F}^{h,\tau}_{k,\varepsilon} \)). Moreover, if assumption (D5) holds, then

\[ (1.1) \quad \hat{U} \leq \hat{U}_{k,\varepsilon} \leq \bar{U} \quad \text{and} \quad \hat{V} \leq \hat{V}_{k,\varepsilon} \leq \bar{V} \quad \text{in } Q_{T}, \]

where \( \hat{U}, \bar{U}, \hat{V}, \) and \( \bar{V} \in \mathbb{R} \) are independent of \( h, \tau, \varepsilon, \) and \( k \). In particular, if \( g_{\varepsilon}^{h} \) and \( f \geq 0 \) then one can take \( \hat{U} = \bar{V} = 0 \).
Proof. The proof is provided in Theorem 4.1 in Part I [3]. □

Lemma 4.1. Under assumption (D4) we have for all \( \varepsilon \in \langle 0, \varepsilon_0 \rangle \), \( h, \tau > 0 \) and \( m = 0 \rightarrow N \) that

\[
\| \hat{u}_{k,\varepsilon}^h - \hat{U}_{k,\varepsilon}^h \|^2_{L^2(k,m\tau)} + \varepsilon |\pi_h[\varphi_e(\hat{u}_{k,\varepsilon}^h) - \varphi_{e}(\hat{U}_{k,\varepsilon}^h)]|_{L^2(k,m\tau)}^2 + \varepsilon \| \hat{\varphi}_{k,\varepsilon} \|^2_{E_1(k,m\tau)} \leq C \tau^2 \left\{ |\partial_t \hat{u}_{k,\varepsilon}^h|_{L^2(Q_T)}^2 + (\tau + k^{-1})^{-1} |\nabla (\partial_t \hat{u}_{k,\varepsilon}^h)|_{L^2(Q_T)}^2 \right. \\
+ |\partial_t \pi_h[\varphi_e(\hat{u}_{k,\varepsilon}^h)]|_{L^2(Q_T)}^2 |+\| \hat{\varphi}_{k,\varepsilon} \|^2_{E_1(k,m\tau)} \right\}. 
\]

Proof. The proof is provided in Lemma 4.1 in Part I [3]. □

Corollary 4.1. Under assumption (D4) we have for all \( \varepsilon \in \langle 0, \varepsilon_0 \rangle \), \( h, \tau > 0 \), and \( m = 0 \rightarrow N \) that

\[
\| \hat{u}_{k,\varepsilon}^h - \hat{U}_{k,\varepsilon}^h \|^2_{L^2(k,m\tau)} + \varepsilon |\pi_h[\varphi_e(\hat{u}_{k,\varepsilon}^h) - \varphi_{e}(\hat{U}_{k,\varepsilon}^h)]|_{L^2(k,m\tau)}^2 + \varepsilon \| \hat{\varphi}_{k,\varepsilon} \|^2_{E_1(k,m\tau)} \leq C(\varepsilon^{-1} + (\tau + k^{-1})^{-1}) \tau^2. 
\]

Proof. The result follows immediately from Lemma 4.1 and (3.4). □

Below we will present an alternative bound to (4.2). First, we prove an analogue of Lemma 4.1. We adopt the proof given for Lemma 2.2 in Part I [3]. We adopt the difference notation \( D_t^+ \hat{u}_{k,\varepsilon}^h \equiv (\hat{u}_{k,\varepsilon}^{h+1} - \hat{u}_{k,\varepsilon}^h)/\tau \), \( D_t^- \hat{u}_{k,\varepsilon}^h \equiv (\hat{u}_{k,\varepsilon}^{h+1} - \hat{u}_{k,\varepsilon}^h)/\tau \), and \( \delta_t^2 = D_t^+ D_t^- \) and note that

\[
\sum_{n=1}^m [(a_n - a_{n-1})a_n] = \frac{1}{2} \left( (a^m)^2 - (a^0)^2 + \sum_{n=1}^m (a_n - a_{n-1})^2 \right). 
\]

Subtracting successive equations in (\( \hat{P}_{k,\varepsilon}^{h,\tau} \)) yields for \( n = 1 \rightarrow N - 1 \)

\[
\langle \delta_t^2 \hat{u}_{k,\varepsilon}^h, \chi \rangle + \langle \nabla D_t^+ \hat{u}_{k,\varepsilon}^h, \nabla \chi \rangle = \langle D_t^+ f^n, \chi \rangle \quad \forall \chi \in S_0^h, \\
\langle \delta_t^2 \hat{u}_{k,\varepsilon}^h, \chi \rangle = k \langle D_t^+ [\varphi_e(\hat{u}_{k,\varepsilon}^h) - \hat{u}_{k,\varepsilon}^h, \chi \rangle \quad \forall \chi \in S^h, 
\]

and hence

\[
\langle \hat{u}_{k,\varepsilon}^h - \hat{u}_{k,\varepsilon}^h, D_t^+ [\hat{u}_{k,\varepsilon}^h, \varphi_e(\hat{u}_{k,\varepsilon}^h)], \chi \rangle + \langle k^{-1} \nabla D_t^+ \hat{u}_{k,\varepsilon}^h + \nabla \hat{u}_{k,\varepsilon}^{h+1}, \nabla \chi \rangle \\
= \langle f^{n+1} + k^{-1} D_t^+ f^n, \chi \rangle \quad \forall \chi \in S^h. 
\]

Choosing \( \chi = D_t^+ \hat{u}_{k,\varepsilon}^h \) in (4.6) and summing from \( n = 1 \rightarrow m \), we note (4.4) yields for \( m = 1 \rightarrow N - 1 \) that

\[
\frac{1}{2}k^{-1} \left[ D_t^+ \hat{u}_{k,\varepsilon}^{h,m} \right]^2_{h} + m \sum_{n=1}^m |\delta_t^2 \hat{u}_{k,\varepsilon}^h|^2_{h} - m \sum_{n=1}^m |\delta_t^2 \hat{u}_{k,\varepsilon}^h|^2_{h} 
\]
Combining (4.11) and (4.12) and noting (3.3a) yields the desired result (4.3).

Choosing $\chi (4.9)$ and hence

$$\hat{D}_t [\varphi (\hat{u}_{k,\varepsilon}^h)], \hat{D}_t \hat{u}_{k,\varepsilon}^h = \tau \sum_{n=1}^{m} \langle f^{n+1} + k^{-1} D_t f^n, D_t \hat{u}_{k,\varepsilon}^h \rangle^h$$

and hence

$$k^{-1} |D_t \hat{u}_{k,\varepsilon}^h|^2_h + \tau \sum_{n=1}^{m} |D_t \hat{u}_{k,\varepsilon}^h|^2_h + |D_t [\varphi (\hat{u}_{k,\varepsilon}^h)]|^2_h + 2 \tau |D_t [\varphi (\hat{u}_{k,\varepsilon}^h)]| \hat{D}_t \hat{u}_{k,\varepsilon}^h$$

$$\leq (\tau + k^{-1}) |D_t \hat{u}_{k,\varepsilon}^h|^2_h + 2 \tau |D_t [\varphi (\hat{u}_{k,\varepsilon}^h)]| \hat{D}_t \hat{u}_{k,\varepsilon}^h$$

(4.7)

Next we note from the first equations in (\hat{P}_{k,\varepsilon}) and the initial conditions that

(4.8a) \hspace{1cm} (1 + k\tau) \langle D_t \hat{u}_{k,\varepsilon}^h, \chi \rangle^h = k\tau \langle D_t [\varphi (\hat{u}_{k,\varepsilon}^h)], \chi \rangle^h \forall \chi \in S^h$

and hence

(4.8b) \hspace{1cm} \langle D_t \hat{u}_{k,\varepsilon}^h, \chi \rangle^h + \frac{k\tau}{1 + k\tau} \langle D_t [\varphi (\hat{u}_{k,\varepsilon}^h)], \chi \rangle^h + \langle \nabla \hat{u}_{k,\varepsilon}^h, \nabla \chi \rangle = \langle f, \chi \rangle^h \forall \chi \in S^h.$

In addition it follows from (2.3a) that for $n = 0 \rightarrow N - 1$

(4.9) \hspace{1cm} M^{-1} e \langle D_t [\varphi (\hat{u}_{k,\varepsilon}^h)], \chi \rangle^h \leq \langle D_t [\varphi (\hat{u}_{k,\varepsilon}^h)], D_t \hat{u}_{k,\varepsilon}^h \rangle^h.$

Choosing $\chi \equiv D_t \hat{u}_{k,\varepsilon}^h$ in (4.8b) and noting (4.9) and Lemma 3.5 yield that

$$\tau |D_t \hat{u}_{k,\varepsilon}^h|^2_h + 2 \tau \frac{k\tau}{1 + k\tau} \langle D_t [\varphi (\hat{u}_{k,\varepsilon}^h)], D_t \hat{u}_{k,\varepsilon}^h \rangle^h + |\nabla \hat{u}_{k,\varepsilon}^h|^2_h \lesssim \leq 1 + (k\tau)^{-1} + \tau \sum_{n=1}^{N} |f^{n+1} + k^{-1} D_t f^n|^2_h \leq C \left[ 1 + (k\tau)^{-1} \right].$$

From (4.7), (4.9), (4.10), and the assumptions on $f$ it follows that

(4.11)

Choosing $\chi \equiv D_t \hat{u}_{k,\varepsilon}^h$ in (4.5b) and summing from $n = 1 \rightarrow m$, we note (4.4) yields for $m = 1 \rightarrow N - 1$ that

$$k^{-1} |D_t \hat{u}_{k,\varepsilon}^h|^2_h + \tau \sum_{n=1}^{m} |D_t \hat{u}_{k,\varepsilon}^h|^2_h \leq k^{-1} |D_t \hat{u}_{k,\varepsilon}^h|^2_h + \tau \sum_{n=1}^{m} |D_t [\varphi (\hat{u}_{k,\varepsilon}^h)]|^2_h.$$

Hence noting (4.8a) we obtain for $m = 1 \rightarrow N - 1$ that

(4.12) \hspace{1cm} k^{-1} |D_t \hat{u}_{k,\varepsilon}^h|^2_h + \tau \sum_{n=1}^{m} |D_t \hat{u}_{k,\varepsilon}^h|^2_h \leq \tau \sum_{n=1}^{m} |D_t [\varphi (\hat{u}_{k,\varepsilon}^h)]|^2_h.$$

Combining (4.11) and (4.12) and noting (3.3a) yields the desired result (4.3).
LEMMA 4.3. Under assumption (D5) we have for all \( \varepsilon \in (0, \varepsilon_0] \), \( h, \tau > 0 \) and for \( m = 0 \to N \) that
\[
\|\hat{u}_{k,\varepsilon}^\varepsilon - \hat{U}_{k,\varepsilon}^\varepsilon\|_{L^2(Q_{m\tau})}^2 + \varepsilon |\nu_k[\varphi_{\varepsilon}(\hat{u}_{k,\varepsilon}^\varepsilon) - \varphi_{\varepsilon}(\hat{U}_{k,\varepsilon}^\varepsilon)]|_{L^2(Q_{m\tau})}^2 + \varepsilon \|\hat{v}_{k,\varepsilon}^\varepsilon - \hat{V}_{k,\varepsilon}^\varepsilon\|_{E_1(k,m\tau)}^2 \\
\leq C(\varepsilon^{-1}\tau^2 + \varepsilon^{-1}k^{-1}\tau + k^{-2}).
\]
(4.13)

Proof. Let \( E_u \equiv \hat{u}_{k,\varepsilon}^\varepsilon - \hat{U}_{k,\varepsilon}^\varepsilon \), \( E_v \equiv \hat{v}_{k,\varepsilon}^\varepsilon - \hat{V}_{k,\varepsilon}^\varepsilon \), \( E_v^\varepsilon \equiv \hat{v}_{k,\varepsilon}^\varepsilon - \hat{V}_{k,\varepsilon}^\varepsilon \), and \( E_f \equiv f - \hat{f} \). First, we note that
\[
|\hat{U}_{k,\varepsilon}^L - \hat{U}_{k,\varepsilon}^\varepsilon|_{L^2(Q_T)}^2 = \sum_{n=1}^N \int_{(n-1)\tau}^{n\tau} (n\tau - t)^2 |\partial_t \hat{U}_{k,\varepsilon}^L|^2 dt \leq \tau^2 |\partial_t \hat{U}_{k,\varepsilon}^L|^2_{L^2(Q_T)}
\]
and the equivalent result with \( U \) replaced by \( V \). Hence it follows from (4.3) that
\[
|\hat{U}_{k,\varepsilon}^L - \hat{U}_{k,\varepsilon}^\varepsilon|_{L^2(Q_T)}^2 + \varepsilon |\hat{V}_{k,\varepsilon}^L - \hat{V}_{k,\varepsilon}^\varepsilon|_{L^2(Q_T)}^2 \leq C(\tau^2 + k^{-1}\tau).
\]
(4.14b)

Similarly we have \( \|E_f\|_{L^2(Q_T)}^2 \leq C\tau^2 \).

It follows from (\( \hat{P}_{k,\varepsilon}^\varepsilon \)) and (\( \hat{P}_{k,\varepsilon}^h \)) that \( E_u^\varepsilon(\cdot,0) = 0 \), \( E_v^\varepsilon(\cdot,0) = 0 \), and for almost every \( t \in (0,T) \)
\[
\langle \partial_t E_u^L + \partial_t E_v^L, \chi^h \rangle + \langle \nabla E_u, \nabla \chi \rangle = \langle E_f, \chi \rangle^h \quad \forall \chi \in S^h,
\]
(4.15a)
\[
\langle \partial_t E_v^L, \chi^h \rangle = k(\varphi_{\varepsilon}(\hat{u}_{k,\varepsilon}^\varepsilon) - \varphi_{\varepsilon}(\hat{U}_{k,\varepsilon}^\varepsilon)) - E_v, \chi \rangle^h \quad \forall \chi \in S^h.
\]
(4.15b)

Choosing \( \chi \equiv \int_0^t E_u(\cdot,\sigma) \) \( d\sigma \) in (4.15a) and integrating over \((0,t)\) in time, where \( s \) is the integration variable in time, yields that
\[
\int_0^t |E_u(\cdot,s)|_{L^2(\Omega)}^2 \, ds + \frac{1}{2} |\nabla \int_0^t E_u(\cdot,s) \, ds|_{L^2(\Omega)}^2
\]
(4.16)
\[= \int_0^t \left( \langle [\hat{U}_{k,\varepsilon}^L - \hat{U}_{k,\varepsilon}^\varepsilon - E_v^L(\cdot,s)], \partial_s E_u^L(\cdot,s) \rangle^h + \langle [E_f - \partial_s E_v^L(\cdot,s), E_u(\cdot,s)] \rangle^h \right) \, ds.
\]
Similarly, choosing \( \chi \equiv E_u \) in (4.15a) yields that
\[
\frac{1}{2} |E_u^L(t)|_{L^2(\Omega)}^2 + \int_0^t \langle \nabla E_u(\cdot,s) \rangle_{L^2(\Omega)}^2 \, ds
\]
(4.17)
\[= \int_0^t \left( \langle [\hat{U}_{k,\varepsilon}^L - \hat{U}_{k,\varepsilon}^\varepsilon], \partial_s E_u^L(\cdot,s) \rangle^h + \langle [E_f - \partial_s E_v^L(\cdot,s), E_u(\cdot,s)] \rangle^h \right) \, ds.
\]
From (4.16), (4.17), (4.15b), (3.4), (4.3), and (4.14b) it follows that
\[
\int_0^t |E_u(\cdot,s)|_{L^2(\Omega)}^2 \, ds + \frac{1}{2} |\nabla \int_0^t E_u(\cdot,s) \, ds|_{L^2(\Omega)}^2 \leq
\]
\[+ k^{-1} \left( \frac{1}{2} |E_u^\varepsilon(\cdot,t)|_{L^2(\Omega)}^2 + \int_0^t \langle \nabla E_u(\cdot,s) \rangle_{L^2(\Omega)}^2 \, ds \right)
\]
\[+ \int_0^t \left( \varphi_{\varepsilon}(\hat{u}_{k,\varepsilon}^\varepsilon(\cdot,s)) - \varphi_{\varepsilon}(\hat{U}_{k,\varepsilon}^\varepsilon(\cdot,s)), E_u(\cdot,s) \right)^h \, ds
\]
\[= \int_0^t \left( \langle [\hat{U}_{k,\varepsilon}^L - \hat{U}_{k,\varepsilon}^\varepsilon], \hat{V}_{k,\varepsilon}^L - \hat{V}_{k,\varepsilon}^\varepsilon \rangle(\cdot,s), E_u(\cdot,s) \right)^h \, ds.
\]
The desired result for \( \tau \) in (4.13) then follows from (4.18), (3.31), and (3.32). Moreover, under assumption (D5) hold. Then for the stated choices of \( \varepsilon \) and \( k \), we have that the error bounds (3.31) and (3.32) hold for \( t = m\tau \), \( m = 0 \rightarrow N \), with \( \hat{u}^{k}_{h,\varepsilon} \) replaced by \( \hat{U}_{k,\varepsilon} \) with \( \tau \leq C k^{-1} = Ch \).

Proof. These results follow from (4.13), (3.22), and (3.23). We note that using (4.2) in place of (4.13) leads to a more restrictive bound on \( \tau \).

Finally we extend the above results to the problem 

\[
(\hat{P}^{h}_{\varepsilon}) 
\] 

For \( n = 1 \rightarrow N \) find \( \hat{u}^{h,n}_{\varepsilon} \in S^{h}_{0} \) such that 

\[
\tau^{-1} \langle (\hat{u}^{h,n}_{\varepsilon} - \hat{u}^{h,n-1}_{\varepsilon}) + [\varphi_{\varepsilon}(\hat{u}^{h,n}_{\varepsilon}) - \varphi_{\varepsilon}(\hat{u}^{h,n-1}_{\varepsilon})], \chi \rangle^{h} + \langle \nabla \hat{u}^{h,n}_{\varepsilon}, \nabla \chi \rangle = (f^{n}, \chi)_{h} \quad \forall \chi \in S^{h}_{0}, 
\]

\[
\hat{u}^{h,0}(\cdot) = g^{h}(\cdot). 
\]

**Theorem 4.3.** Let assumption (D4) hold. Then for all \( \varepsilon \in (0, \varepsilon_{0}] \), \( h \), and \( \tau > 0 \) there exists a unique solution \( \hat{U}_{\varepsilon} \) to \((\hat{P}^{h,\tau}_{\varepsilon})\). In addition for \( m = 0 \rightarrow N \) we have that

\[
\|\hat{U}_{\varepsilon} - \hat{U}_{k,\varepsilon}\|_{E_{2}(\infty, m\tau)} + \varepsilon \|\pi_{h}[\varphi_{\varepsilon}(\hat{U}_{\varepsilon}) - \varphi_{\varepsilon}(\hat{U}_{k,\varepsilon})]\|_{L^{2}(Q_{m\tau})}^{2} + \varepsilon \|\pi_{h}[\varphi_{\varepsilon}(\hat{U}_{\varepsilon})] - \hat{V}_{k,\varepsilon}\|_{L^{2}(Q_{m\tau})}^{2} 
\]

\[
\leq C k^{-2} |\partial_{\varepsilon} \hat{V}_{k,\varepsilon}|_{L^{2}(Q_{\tau})}^{2} \leq C \varepsilon^{-1} k^{-2} [1 + (k\tau)^{-1}]. 
\]

Moreover, under assumption (D5) we have the following:

(i) the first bound in (4.1) holds true for \( \hat{U}_{\varepsilon} \). In particular, if \( g^{h}_{\varepsilon} \) and \( f \geq 0 \) then \( \hat{U}_{\varepsilon} \geq 0 \) in \( Q_{\tau} \);
(ii) on choosing \( \tau = Ch \), the error bounds (3.31) and (3.32) hold for \( t = m \tau \), \( m = 0 \to N \), under the stated choices of \( \varepsilon \) with \( \hat{u}_{k,\varepsilon}^h \) replaced by \( \hat{U}_\varepsilon \).

Furthermore, under assumption (D6) we have that the following error bounds hold with \( \hat{u}_{k,\varepsilon}^h \) replaced by \( \hat{U}_\varepsilon \) for \( t = m \tau \), \( m = 0 \to N \), under the stated choices of \( \varepsilon \):

(i) (3.33) for \( p \in (1/2, 1] \) and \( \tau = C \{ h^2 \ln(1/h) \}^{2/(5-2p)} \);

(ii) (3.34) for \( p \in (1/3, 1] \) and \( \tau = C \{ h^2 \ln(1/h) \}^{2(3-p)/(13-7p)} \).

Proof. Existence and uniqueness of a solution to \( \hat{P}_{h,\tau}^\varepsilon \) follows as in the proof of Theorem 4.1 of Part I [3]. The first inequality in (4.20) is a discrete analogue of the first inequality in (2.8b) and is proved in a similar way. The second inequality in (4.20) follows from (4.3). The first bound in (4.1) follows from (4.20) and from the equivalence of norms on \( S_h^\varepsilon \). The above error bounds follow by combining (4.20) with Theorem 4.2. \( \Box \)

As stated in sections 1 and 2, problem (P) is equivalent to the generalized porous medium equation, whose finite element approximation by \( \hat{P}_{k,\varepsilon}^h \) is analyzed in [11]. There the error bounds (3.31a) and (3.32a) for \( \hat{u}_{k,\varepsilon}^h \) replaced by \( \hat{U}_\varepsilon \) are proved under the same choices of \( \varepsilon \), but with \( \tau = Ch^{1+p} \) and \( \tau = Ch^{2/(3-p)} \), respectively. Therefore, Theorem 4.3 above improves on these results as we require only \( \tau = Ch \). As stated previously we have assumed that the mesh is (weakly) acute, whereas they do not. Furthermore, under additional assumptions we have the improved error bounds (3.33) and (3.34).

Remark 4.1. For the numerical computations it seems to be advantageous to use the unrelaxed form \( \hat{P}_{k,\varepsilon}^{h,\tau} \) instead of the relaxed form \( \hat{P}_{k,\varepsilon}^{h,\tau} \), as the unknown \( v \) does not appear. However, note that for a fixed time level \( n \) the set of equations in \( \hat{P}_{k,\varepsilon}^{h,\tau} \) is equivalent to

\[
\tau^{-1} \left( \hat{u}_{k,\varepsilon}^{h,n} - \hat{u}_{k,\varepsilon}^{h,n-1} \right) + \frac{k \tau}{1 + k \tau} (\varphi(\hat{u}_{k,\varepsilon}^{h,n}) - \hat{v}_{k,\varepsilon}^{h,n-1}, \chi) + \left( \nabla \hat{u}_{k,\varepsilon}^{h,n}, \nabla \chi \right) + \left( f^n, \chi \right)^h \forall \chi \in S_0^h
\]

This means that the computational complexity of both variants is nearly identical.

5. A numerical experiment. Finally we discuss some numerical experiments. As an example we take a transformation of the well-known Barenblatt solution,

\[
w(x, t) = (t + 1)^{-\frac{1}{m+1}} \left[ 1 - \frac{m - 1}{2m(m + 1)} \left( \frac{x}{(t + 1)^{\frac{1}{m+1}}} \right)^2 \right]^{\frac{1}{m-1}},
\]

of the classical porous medium equation (2.15) with \( \beta(w) \equiv \text{sgn}(w)|w|^m \), \( m > 1 \), and \( f \equiv 0 \) in one space dimension; see, e.g., [1]. Then \( u \equiv w^m \) can be interpreted as a solution of problem (P) with

\[
\varphi(s) \equiv \begin{cases} \lfloor m s^{1/m} \rfloor - s & \text{if } s \leq 1, \\ m - 1 & \text{if } s \geq 1 \end{cases}
\]

and for the appropriate choice of \( Q_T \) and initial condition \( g \) if \( -\Delta u \) is replaced by \( -m \Delta u \) or, equivalently, a time scaling is performed. As one can take \( u = 0 \) and
\[ E^h = \left( \frac{1}{N} \sum_{n=1}^{N} |u(\cdot, n\tau) - \hat{u}_h^n(\cdot)|^2 \right)^{1/2}, \]

where \( N\tau \leq T < (N+1)\tau \). It follows from (3.3a), a bound similar to (4.14a), (4.3), and (4.14a) for \( k = \infty \) and noting that \( u \in C^\nu(\Omega_T) \), where \( \nu = \min\{1, 1-p\} \), that

\[
\begin{align*}
[E^h]^2 &\leq C\rho u - \hat{U}_\varepsilon^2|_{L^2(\Omega_T)} + C\tau^2|\partial_t(\pi_h u - \hat{U}_\varepsilon)|_{L^2(\Omega_T)}^2 \\
&\leq C|u - \hat{U}_\varepsilon|^2_{L^2(\Omega_T)} + C|\pi_h u - \hat{U}_\varepsilon|^2_{L^2(\Omega_T)} + C\tau^2 \\
&\leq C|u - \hat{U}_\varepsilon|^2_{L^2(\Omega_T)} + C\rho^2 + C\tau^2.
\end{align*}
\]

Noting Theorem 4.3, we see that the approximation \( E^h \) is of sufficient accuracy.

We estimate the rate of convergence of \( E^h \) by setting

\[ \alpha^h = \frac{\ln[E^{2h}/E^h]}{\ln 2}. \]

Inspecting Tables 5.1 and 5.2 we see that the actual convergence rates for the approximations \( \hat{U}_\varepsilon \) are better than that predicted by the theory, which appear in the tables next to the corresponding assumption, i.e., \( \alpha^h = 2/(3-p) \) and \( 8/(13-7p) \) under assumptions (D5) and (D6), respectively (the latter being restricted to \( p > 1/3 \)). It should be noted though that the actual convergence rates have not yet settled down, especially for smaller values of \( p \).
Table 5.2.

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
k^{-1} & p = 0.7 & & & p = 0.9 & & \\
& (D5) & 0.87 & (D6) & 0.99 & (D5) & 0.95 & (D6) \\
\hline
E^b \times 10^4 & \alpha^b & E^b \times 10^4 & \alpha^b & E^b \times 10^4 & \alpha^b & E^b \times 10^4 & \alpha^b \\
\hline
4 & 35.34 & 30.85 & & 33.27 & 26.15 & & \\
8 & 15.42 & 1.20 & 12.19 & 1.34 & 14.31 & 1.22 & 9.41 & 1.47 \\
16 & 7.15 & 1.11 & 5.08 & 1.26 & 6.58 & 1.12 & 3.55 & 1.41 \\
32 & 3.44 & 1.06 & 2.20 & 1.20 & 3.14 & 1.07 & 1.39 & 1.35 \\
\hline
\end{array}
\]

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REFERENCES