Rudolf Gorenflo
Karl-Heinz Hoffmann

Applied nonlinear Functional analysis
Variational Methods and ill-posed Problems


STABILITY THEOREMS FOR GENERAL FREE BOUNDARY PROBLEMS
OF THE STEFAN TYPE AND APPLICATIONS

Peter Knabner
Naturwissenschaftliche Fakultät der
Universität Augsburg
8900 Augsburg

1. Introduction
Free-boundary problems of Stefan type in one space dimension have been widely investigated during the last years. Especially the question of stability of the free boundary, i.e. its continuous dependence on the data of the problem, has been treated and answered affirmative. This has been done for the classical Stefan problem in Cannon/Hill [1], Cannon/Douglas [2] and Cannon/Primicerio [3], and for a general Stefan-like free-boundary problem in Fasano/Primicerio [4]-[6].

The final aim of this paper is the application of such stability theorems in the investigation of numerical schemes for Stefan problems and, as the main subject, for a related approximation problem, sometimes called the "inverse Stefan problem" (for exact definition see section 5). Unfortunately the above mentioned results are unsatisfactory or even not applicable for the applications we have in mind:

(1) for a posteriori estimates: The stability estimate should be as sharp as possible. Recently Reemtsen/Lozano [8] have used the techniques, on which all the above mentioned results are founded, with special regard to this aspect to get an a posteriori estimate for a special numerical scheme. In
section 4 we compare their result with a corollary from our stability theorem.

(2) for order of convergence estimates: For the "inverse Stefan problem" we will consider a scheme with varying data in an unbounded set. Therefore it is essential that the stability estimate is global with respect to these data. This is not the case for the known results and renders them inapplicable.

Our main result will be a stability theorem with these necessary properties and for a problem general enough to cover all applications we have in mind (especially more general as the classical Stefan problem).

To be specific, we will treat the following one phase Stefan-like problem with linear Neumann boundary conditions (possibly degenerated):

\[ T > 0. \]

\[ (1.1) \quad Lu = u_{xx} - u_t = q \text{ in } D_T(s) = \{(x,t) \mid 0 < t < T, 0 < x < s(t)\} \]

\[ (1.2) \quad u(x,0) = h(x), \quad 0 < x < b = s(0) \text{ (if } b > 0) \]

\[ (1.3) \quad u_x(0,t) = g(t) \]

\[ (1.4) \quad u(s(t),t) = f(s(t),t) \]

\[ (1.5) \quad u_x(s(t),t) = \lambda(s(t),t) \dot{s}(t) + \mu(s(t),t) \]

We will consider only classical solutions, i.e. \((u,s)\) is a solution of (1.1)-(1.5) in \((0,T)\) if

- \(s\) is positive in \((0,T),\)
- \(s\) is continuously differentiable in \((0,T)\) and continuous in \([0,T],\)
- \(u\) is continuous in \(D_T(s)\) except for a finite number of discontinuities at \(x = 0\) and \(t = 0\), where \(\lim \inf u(x,t)\) and \(\lim \sup u(x,t)\) are bounded,
- \(u_x\) is continuous for \(0 < t < T, 0 < x < s(t)\),
- \(u_{xx}, u_t\) are continuous in \(D_T(s)\) and (1.1)-(1.5) are satisfied.

For the convenience of the reader, we use the nomenclature in Fasano/Primicerio [4], [5], to our knowledge the most general investigation of linear one dimensional Stefan problems.

It is possible to prove analogous stability theorems for one phase Stefan-like problems with Dirichlet boundary conditions or for a class of nonlinear boundary conditions covering the most important physical cases as Newton's or Stefan-Boltzmann's law. The treatment of the two phase Stefan problem with Neumann boundary conditions is also possible. We will consider these cases in a subsequent paper.

2. Regularity assumptions for the data

In the following we list the properties, we assume for the data in the rest of the paper:

Given \(T > 0\), set \(\Omega_T = \{(x,t) \mid x > 0, 0 < t < T\}\)

\[ (2.1) \quad q \text{ is locally Hölder continuous in } \bar{\Omega}_T \text{ with respect to } x \text{ and bounded in bounded } M \subset \bar{\Omega}_T. \]

\[ (2.2) \quad g \text{ is piecewise continuous in } [0,T]. \]

\[ (2.3) \quad \lambda, \lambda_x, \lambda_t \text{ are continuous in } \bar{\Omega}_T \text{ and there exist } \lambda', \lambda'' > 0 \text{ such that } \lambda' \leq |\lambda(x,t)| \leq \lambda'' \text{ for all } (x,t) \in \bar{\Omega}_T. \]

\[ (2.4) \quad u \text{ is continuous in } \bar{\Omega}_T. \]
(2.5) \( b \geq 0 \)

If \( b > 0 \), then \( h \) is piecewise continuous in \( [0, b] \).

Because of (2.3) \( \lambda \) has only one sign. We consider only the case of negative \( \lambda \), the other one may be reduced to this one.

3. The stability theorem

3.1 Theorem: Given data sets \((b_1, h_1, q_1, f_1, \lambda_1, v_1)\) with solutions \((u_1, s_1)\) in \((0, T)\), \(i = 1, 2\), \(T_1^* > T, (0, T_1^*)\) denoting the maximal interval of existence of \((u_1, s_1)\). For the data we assume

setting \( \Sigma(t) := \max(s_1(t), s_2(t)) \), \( \sigma(t) := \min(s_1(t), s_2(t)) \):

\[ \lambda_1 = \lambda_2 = \lambda \]
\[ f_1 = f_2 = 0 \]
\[ \hat{x}(t) := \sup((-\lambda \hat{x}(x, t)) \big| x \geq \sigma(t)) \in \mathbb{R} \]

(3.1) For all \( x \in [0, \sigma(0)] \) there is an \( j \in \{1, 2\} \) such that

\[ h_j(x) \geq 0 \]

(3.2) For all \( t \in (0, T) \) there is an \( j \in \{1, 2\} \) such that

\[ g_j(t) \leq 0 \]

(3.3) For all \((x, t) \in D_\xi(t)\) there is an \( j \in \{1, 2\} \) such that

\[ g_j(x, t) \leq 0 \]

Set

\[ H := \max(\|h_1\|_\infty, \|h_2\|_\infty) \]

\[ Q(t) := \sup \{\max(-q_1(x, t), -q_2(x, t)) \big| \sigma(t) \leq x \leq \sigma(t)\} \]

\[ C(t) := \lambda^{-1} \exp(\lambda^{-1}(\int_0^t Q(x) + \hat{x}(x, t) dt)) \]

Then for all \( t \in (0, T) \):

\[ |s_1(t) - s_2(t)| \leq C(t) \left\{ (\lambda^{1/2} + H)|b_1 - b_2| + \int_0^t |h_1(x) - h_2(x)| dx + \int_0^t |g_1(t) - g_2(t)| dt + \int_0^t \max_{i=1,2} \left( \mu_i \hat{s}_i(t) - \mu_i \hat{s}_i(t) \right) dt \right\} \]

3.2 Remark: If we consider the following special case

\[ \lambda_1 \geq 0 \text{ in } \mathbb{R} \]
\[ q_1 = 0 \text{ in } D_T(s_1) \]

\[ \mu_1 \text{ does not depend on } x : \mu_1(x, t) = \mu_1(t) \]

the estimate reduces to

\[ |s_1(t) - s_2(t)| \leq \lambda^{-1}(\lambda^{1/2} + H)|b_1 - b_2| + \int_0^t |h_1(x) - h_2(x)| dx + \int_0^t |g_1(t) - g_2(t)| dt + \int_0^t \mu_1(t) \hat{s}_1(t) dt \]

This means especially for the classical Stefan problem constant 1 in the Lipschitz estimate (compare Cannon/Douglas [2]).

3.3 Remark: Comparing Theorem 3.1 with the results of Pasano/Primicerio ([4] for \( b > 0 \), [5] for \( b = 0 \), results

* \[ y_* := \max(y, 0) \text{ for } y \in \mathbb{R} \]
only formulated) we see that they consider a more general situation: They have no sign conditions as (3.1)-(3.3) and allow also variations of $\lambda$.

On the other hand they need more regularity for $s$, which may be ensured by more regularity for the data as given in section 2 (for example Hölder continuity of $h$ at $x = b$) and, most important, the Lipschitz constant is in every case dependent on $g$ and $\varphi$.

3.4 Proof of Theorem 3.1

(i) Construction of a sequence of comparison data.

Given $\varepsilon > 0$, sufficiently small, then set:

$$\bar{b}_\varepsilon := \max(b_1, b_2) + \varepsilon$$

$$\bar{h}(x) := \max(\bar{h}_1(x), \bar{h}_2(x)) , x \geq 0$$

with $\bar{h}_1(x) := \begin{cases} h_1(x), x < [0, b_1] \\ 0, x > b_1 \end{cases}$ for $b_1 > 0$

and $\bar{h}_1(x) = 0, x \geq 0$, for $b_1 = 0$

$$\bar{q}_\varepsilon(t) := \begin{cases} \min(q_1(t), q_2(t)) - \varepsilon, t \in [0, T] \\ \bar{q}_\varepsilon(T), t > T \end{cases}$$

$$\bar{a}_\varepsilon := \lambda$$

$$\bar{f} := 0$$

(ii) Existence of a solution $(\bar{u}_\varepsilon, \bar{s}_\varepsilon)$ to the comparison data.

The local existence of $(\bar{u}_\varepsilon, \bar{s}_\varepsilon)$ is an immediate consequence of Fasano/Primicerio [4, Th. 4]: The regularity assumptions of this paper are satisfied, especially $\bar{b}_\varepsilon > 0$ and the local Lipschitz continuity of $\bar{h}$ at $x = \bar{b}_\varepsilon$.

We want to show that $(\bar{u}_\varepsilon, \bar{s}_\varepsilon)$ exists in $[0, T]$, i.e. $T^* > T$, $[0, T^*]$ denoting the maximal interval of existence of $(\bar{u}_\varepsilon, \bar{s}_\varepsilon)$.

For this reason we have to collect some properties of $(\bar{u}_\varepsilon, \bar{s}_\varepsilon)$:

We have

$$\bar{h}(x) \geq 0 \text{ for } x \in [0, \bar{b}_\varepsilon]$$

$$\bar{g}_\varepsilon(t) < 0 \text{ for } t > T$$

$$\bar{q}_\varepsilon(x, t) \leq 0 \text{ for } x \geq 0, t \geq 0$$

$$\bar{f} = 0$$

and therefore from the strong maximum principle

$$(3.4) \quad \bar{u}_\varepsilon(x, t) > 0 \text{ for } (x, t) \in D_{T^*}(\bar{s}_\varepsilon)$$
(compare e.g. Sherman [9], Le. 1)

From (3.4) it follows

\[ \bar{u}_{\xi}(\bar{s}_{\xi}(t), t) \leq 0 \text{ for all } t \in (0, T^*_c) \]

and therefore

\[ \bar{s}_{\xi}(t) \geq -\lambda^{-1} M > -\infty \text{ for all } t \in (0, T^*_c) \]

M denoting a bound of |\bar{u}| in \( \mathbb{R}^s \).

Furthermore

\[ b_{\xi} > b \]
\[ h(x) \geq h_1(x) \text{ for } x \in [0, b_1] \]
\[ q_{\xi}(t) < q(t) \text{ for } t \in (0, T) \]
\[ q_{\xi}(x, t) \leq q_{1}(x, t) \text{ for } (x, t) \in D_{\xi}(s_{1}) \]
\[ \bar{u}(t) \geq \bar{u}_{1}(s_{1}(t), t) \text{ for } t \in (0, T) \]

This together with (3.4) implies

\[ s_{1}(t) \leq \bar{s}_{\xi}(t) \text{ for } t \in [0, \min(T^*_1, T^*_c)) \]

(Fasano/Primicerio [4], Th. 9*)

Three cases are possible:

a) \( T^*_c = \infty \)

b) \( T^*_c < \infty \) and \( \liminf_{t \to T^*_c} s(t) = 0 \)

c) \( T^*_c < \infty \), \( \liminf_{t \to T^*_c} s(t) > 0 \) and \( \liminf_{t \to T^*_c} \dot{s}(t) = -\infty \)

(Fasano/Primicerio [4], Th. 8)

Assume, \( T^*_c \leq T \):

Case c) is not possible because of (3.5).

If case b) is valid, we have \( s_{1}(T^*_c) = 0 \) because of (3.7):

This is a contradiction, if \( T^*_c < T \), therefore:

\[ T^*_c \geq T \text{ and } (T^*_c = T \Rightarrow s_{1}(T) = 0 \text{ for } i = 1, 2) \]

(iii) The estimate:

(3.7) implies:

For all \( t \in [0, T] \) there exists \( j \in \{1, 2\} \) such that

\[ |s_{1}(t) - s_{2}(t)| \leq \bar{s}_{\xi}(t) - s_{j}(t) \]

as we may assume for the estimation of \( |s_{1}(t) - s_{2}(t)| \) that \( T^*_c > T \) because of (3.8).

Therefore it suffices to estimate \( \bar{s}_{\xi}(t) - s_{1}(t) \) for \( i = 1, 2, \).

The following integral relation holds for the solution of (1.1)-(1.5) (Fasano/Primicerio [4], (3.6)):

\[ -\Lambda(s(t), t) = -\Lambda(b, 0) - \int_{D_{\xi}(x)} q(x, \tau) dx d\tau + \int_{0}^{b} h(x) dx \]
\[ -\int_{0}^{t} q(t) d\tau - \int_{0}^{t} \Lambda_{\xi}(s(\tau), \tau) d\tau + \int_{0}^{t} u(s(\tau), \tau) d\tau \]
\[ -\int_{0}^{s(t)} u(x, t) dx \text{ for } t \in [0, T] \]

using \( \Lambda(x, t) = \int_{0}^{x} \lambda(\xi, t) d\xi \).

Let \( i \in \{1, 2\} \), \( 0 \leq t \leq T \).

Subtracting this relation für \( (\bar{u}_{\xi}, \bar{s}_{\xi}) \) and \( (u_{i}, s_{i}) \) and using

* This and the following cited theorems deal with Dirichlet boundary conditions, but analogous ones can be proved for Neumann boundary conditions without difficulty.
\[ -\lambda'(\tilde{s}_e(t), t) - \lambda(s_\lambda(t), t) = -\int_{s_\lambda(t)}^{\tilde{s}_e(t)} \lambda(\xi, t) d\xi \geq \lambda' \left( \tilde{s}_e(t) - s_\lambda(t) \right) \]

we get:

\[ \tilde{s}_e(t) - s_\lambda(t) \leq \lambda^{-1} \left( -\lambda(\tilde{b}_e, 0) - \lambda(b, 0) \right) - \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} g_\lambda(x, t) dx \, dt - \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} \tilde{g}_e(x, t) dx \, dt + \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} \tilde{h}(x) - h_\lambda(x) dx + \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} \tilde{g}_e(x, t) dx - \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} \tilde{g}_e(x, t) dx \, dt - \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} \tilde{h}(x) - h_\lambda(x) dx + \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} \tilde{g}_e(x, t) dx - \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} \tilde{g}_e(x, t) dx \, dt - \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} \tilde{h}(x) - h_\lambda(x) dx + \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} \tilde{g}_e(x, t) dx - \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} \tilde{g}_e(x, t) dx \, dt \]

\[ = : \lambda^{-1} \sum_{j=1}^{10} A_j(t) \]

Estimating the terms in detail, we get with \( j = 3-i \):

\[ A_1^{(i)} \leq \lambda''(\tilde{b}_e - b_\lambda) \leq \lambda''(\|b_1 - b_2\| + \varepsilon) \]

\[ A_2^{(i)} = \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} \left( (q_1 - q_3)(x, t) \right) dx \, dt \leq \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} |q_1(x, t) - q_2(x, t)| dx \, dt \]

\[ A_3^{(i)} = \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} \left( \left( (q_1 - q_3)(x, t) \right) \right) dx \, dt \leq \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} |q_1(x, t) - q_2(x, t)| dx \, dt \]

\[ \lambda_3^{(i)} = \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} \left( (q_1 - q_3)(x, t) \right) dx \, dt \leq \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} |q_1(x, t) - q_2(x, t)| dx \, dt \]

\[ \lambda_4^{(i)} = \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} \left( (q_1 - q_3)(x, t) \right) dx \, dt \leq \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} |q_1(x, t) - q_2(x, t)| dx \, dt \]

\[ \lambda_5^{(i)} = \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} \left( (q_1 - q_3)(x, t) \right) dx \, dt \leq \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} |q_1(x, t) - q_2(x, t)| dx \, dt \]

\[ \lambda_6^{(i)} = \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} \left( (q_1 - q_3)(x, t) \right) dx \, dt \leq \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} |q_1(x, t) - q_2(x, t)| dx \, dt \]

\[ \lambda_7^{(i)} = \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} \left( (q_1 - q_3)(x, t) \right) dx \, dt \leq \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} |q_1(x, t) - q_2(x, t)| dx \, dt \]

\[ \lambda_8^{(i)} = \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} \left( (q_1 - q_3)(x, t) \right) dx \, dt \leq \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} |q_1(x, t) - q_2(x, t)| dx \, dt \]

\[ \lambda_9^{(i)} = \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} \left( (q_1 - q_3)(x, t) \right) dx \, dt \leq \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} |q_1(x, t) - q_2(x, t)| dx \, dt \]

\[ \lambda_{10}^{(i)} = \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} \left( (q_1 - q_3)(x, t) \right) dx \, dt \leq \int_{D_\lambda(s_\lambda)}^{\tilde{b}_e(x, t)} |q_1(x, t) - q_2(x, t)| dx \, dt \]
\( A_{9}^{(1)} = \int_{0}^{t} \left( u_{j}(s_{1}(\tau), \tau) - u_{1}(s_{1}(\tau), \tau) \right) d\tau \)

Concerning \( A_{9}^{(1)} \), set
\[ v_{1}(x, \tau) = \bar{u}_{\varepsilon}(x, \tau) - u_{1}(x, \tau) \text{ for } 0 \leq \tau \leq T, 0 \leq x \leq s_{1}(\tau), \]
then according to (3.6) and (3.4):
\[
\begin{align*}
&v_{1}(x, 0) \geq 0 \text{ for } 0 \leq x \leq b_{1}
&v_{1}(0, \tau) \leq 0 \\
&v_{1}(s_{1}(\tau), \tau) \geq 0 \\
&\text{for } 0 < \tau \leq T
\end{align*}
\]
The maximum principle implies \( v_{1}(x, \tau) \geq 0 \text{ for } 0 < \tau \leq T, 0 < x < s_{1}(\tau) \) and therefore
\[ A_{9}^{(1)} = -\int_{0}^{s_{1}(t)} v_{1}(x, t) dx \leq 0 \]
Because of (3.4) we have
\[ \bar{u}_{\varepsilon}(x, \tau) \geq 0 \text{ for } 0 < \tau \leq T, 0 < x < s_{\varepsilon}(\tau) \]
i.e.
\[ A_{10}^{(1)} = -\int_{s_{1}(t)}^{s_{\varepsilon}(t)} \bar{u}_{\varepsilon}(x, t) dx \leq 0 \]
Using these results, we get:
\( (3.10) \quad \delta_{1}(t) \leq \lambda^{-1} \left[ \sum_{j=1}^{8} A_{j}^{(1)} + \bar{Q} t \varepsilon \right] + \int_{0}^{t} \left( Q(\tau) + \bar{\kappa}(\tau) \right) \delta_{1}(\tau) d\tau \)

As the expression in square brackets is non-decreasing in \( t \), as seen from above, the lemma of Gronwall is applicable and we get:
\[ \delta_{1}(t) \leq \exp(\lambda^{-1} \int_{0}^{t} \left( Q(\tau) + \bar{\kappa}(\tau) \right) d\tau) \left( \sum_{j=3, 7} A_{j}^{(1)} + \bar{Q} t \varepsilon \right) \]

From (3.9) and the above estimates we have
\[ |s_{1}(t) - s_{2}(t)| \leq C(t) \left( (\lambda^{-1} + \bar{Q} t \varepsilon) |b_{1} - b_{2}| + \int_{0}^{b(0)} |h_{1}(x) - h_{2}(x)| dx \right. \]
\[ + \int_{0}^{t} |g_{1}(\tau) - g_{2}(\tau)| d\tau + \int_{D_{c}} |q_{1}(x, \tau) - q_{2}(x, \tau)| dx d\tau \]
\[ + \max_{i=1, 2} \left( \int_{0}^{t} \left( u_{j}(s_{i}(\tau), \tau) - u_{1}(s_{i}(\tau), \tau) \right) d\tau \right) \text{ for } 0 \leq t \leq T \]
\[ c \leq 0 \text{ gives the assertion.} \]

3.5 Remark: The proof shows that some of the estimates may be sharpened, for example instead of
\[ " H |b_{1} - b_{2}| + \int_{0}^{c(0)} |h_{1}(x) - h_{2}(x)| dx " \]
\[ " \int_{0}^{c(0)} |h_{1}(x) - h_{2}(x)| dx " \]
is possible.

We note a sharpening of the estimate for a special case, which will be useful in the applications.

3.7 Corollary: If the assumptions of Theorem 3.1 are valid and additionally \( q_{1} = 0 \) the constant can be sharpened to
\[ C(t) = \lambda^{-1} \exp(\lambda^{-1} \int_{0}^{t} \bar{\kappa}(\tau) d\tau) \]
Proof: The proof of Theorem 3.1 is modified in the following way:

Noting that

\[ A_2^{(1)} + A_3^{(1)} \leq \int_{D_t(s_1)} (-q_2(x,\tau)) \, dx \, d\tau + \tilde{\eta}_t \epsilon \]

\[ A_2^{(2)} + A_3^{(2)} \leq \int_{D_t(s_2)} (q_2(x,\tau)) \, dx \, d\tau + \]

\[ + \int_{D_t(s_1) \setminus D_t(s_2)} (-q_2(x,\tau)) \, dx \, d\tau + \tilde{\eta}_t \epsilon \]

i.e.

\[ A_2^{(1)} + A_3^{(1)} \leq \int_{D_t(s)} |q_2(x,\tau)| \, dx \, d\tau + \tilde{\eta}_t \epsilon \quad \text{for } i = 1, 2 \]

we derive instead of (3.10):

\[ \delta_j(t) \leq \lambda^{i-1} \left( \sum_{j=1}^{8} A_j^{(i)} + \tilde{\eta}_t \epsilon \right) + \int_0^t \tilde{\lambda}(\tau) \delta_j(\tau) \, d\tau \]

Now going on as above, we get the assertion.

3.8 Remark: Up to now we did not treat the case \( f_1 \not= 0 \). If \( f_1 \) is smooth enough, we can use the transformation \( u_1 = u_1 - f_1 \) and apply the results for \( f_1 = 0 \).

4. A posteriori estimates via the stability theorem

We demonstrate the derivation of a posteriori estimates from the stability theorem using a numerical scheme, recently considered by Reemtsen/Lozano [6], as an example.

The problem, they treat, is the classical degenerate one phase Stefan problem with sign condition, i.e. in the notation of (1.1)-(1.5)

\[ b = 0 \]
\[ q = u = f = 0 \]
\[ \lambda = -1 \]

\( g \) is continuous in \([0,T]\), \( g(t) \leq 0 \) for \( t \in [0,T] \), \( g(0) < 0 \).

The investigated numerical scheme is a technique minimizing defects:

An approximation \( (\tilde{u}, \tilde{s}) \) to the solution \((u,s)\) is obtained in the following way:

\( \tilde{u} \) is taken from a finite-dimensional space \( X_1 \), all elements \( v \) of which satisfy \( Lv = 0 \) (i.e. (1.1)) in a sufficiently large domain (including \( D_T(s) \)), \( \tilde{s} \) is taken from a finite-dimensional space \( Y_p \) in such a way that they solve the following approximation problem:

Minimize over \( X_1 \times Y_p \)

\[ \max\{ \| \rho_1(u) \|_T, \| \rho_2(u,s) \|_T, \| \rho_3(u,s) \|_T \} \]

using

\[ \rho_1(u)(t) = u_x(0,t) - g(t) \]
\[ \rho_2(u,s)(t) = u(s(t),t) \]
\[ \rho_3(u,s)(t) = u_x(s(t),t) + \dot{s}(t) \]

and

\[ \| f \|_{\tilde{L}} = \max\{ \| f(t) \|_T \in [0,T] \} \]

Using the techniques, on which most known stability estimates are founded, and modifying them appropriately to get sharp estimates, Reemtsen/Lozano derive the following
Theorem: (Theorem 2 in [6]).

Assume

\( \tilde{u} \) is continuous in \( D^2_\alpha(s) \), \( \rho_1 \) is continuous in \([0,T]\)

\( \tilde{s} \) is monotone nondecreasing in \([0,T]\)

\( g(t) + \rho_1(t) s(t) \leq 0 \) for \( t \in [0,T] \)

\[ \| \rho_2 \| T < 1 \]

Setting

\( N \) for the Lipschitz constant of \( \tilde{s} \), \( M = \max(\| g \| T^n) \)

\[ Q(t) = (1 + \tau^{-1/2} M t^{1/2}) \exp(M^2 t/4) \]

\[ C(t) = \pi^{-1/2} Q(t) \left( \| g \| T + \| \rho_1 \| T \right)^2 (1 - \| \rho_2 \| T)^{-1} \]

then

\[ \| \tilde{s} - \tilde{s} \| T \leq (1 + 2C(t) t^{1/2}) \exp(2C^2(t) t) (1 - \| \rho_2 \| T)^{-1} \]

\[ + \left( (2 \| \rho_1 \| T + \| \rho_2 \| T) \| N \| \rho_3 \| T \right) t^* + \]

\[ + 2 \pi^{-1/2} Q(t) t^{1/2} \| \rho_2 \| T \]

The approximate solution \((\tilde{u}, \tilde{s})\) can interpreted as solution of (1.1)-(1.5) for the following data set

\( \tilde{b} = 0, \tilde{q} = g + \rho_1, \tilde{s} = 0, \tilde{p} = \rho_2, \tilde{x} = -1, \tilde{w} = \rho_3 \).

Therefore it is also possible to apply the stability theorem 3.1:

4.1 Corollary: Assume

\( \rho_3 \) is continuous in \([0,T]\)

\( \hat{\nu}_2 \) exists and is bounded in \([0,T]\)

Then

\[ \| \tilde{s} - \tilde{s} \| T \leq \int_0^T |\rho_1(t)| dt + \int_0^T |\rho_3(t)| dt + \]

\[ + \int_0^T \max(s(t), \tilde{s}(t)) |\hat{\nu}_2(t)| dt \]

for all \( t \in [0,T] \)

Proof: we have the data sets

for \((u, s): b_1 = 0, g_1 = g, q_1 = 0, f_1 = 0, \lambda_1 = -1, \mu_1 = 0 \)

for \((\tilde{u}, \tilde{s}): \tilde{b}_2 = 0, \tilde{g}_2 = g + \rho_1, \tilde{q}_2 = 0, \tilde{f}_2 = \rho_2, \lambda_2 = -1, \tilde{w}_2 = \rho_3 \)

Using the transformation \( \tilde{u} \mapsto \tilde{u} - \hat{\nu}_2 \) the second data set changes to

\[ b_2 = 0, g_2 = g + \rho_1, q_2 = \rho_2, f_2 = 0, \lambda_2 = -1, \mu_2 = \rho_3 \]

Now all assumptions of Corollary 3.7 are fulfilled. (If \( T_2 = T \) consider the problem in \([0, T - \epsilon], \epsilon > 0 \) sufficiently small and then set \( \epsilon \to 0 \).

Because of \( \tilde{x}(t) = 0, \lambda' = 1 \) we have \( C(t) = 1 \). Using

\[ \int \int_{D_\epsilon(T)} |\hat{\nu}_2(t)| dt = \int_0^T \max(s(t), \tilde{s}(t)) |\hat{\nu}_2(t)| dt \]

we get the assertion.

4.2 Remark: 1) The regularity assumption above are only of theoretical nature: Reemtsen/Lozano [6] e.g. use heat polynomials for \( X_1 \), and polynomials for \( Y_p \), where everything is satisfied.

2) There is an explicit formula for \( \hat{\nu}_2 \) involving basis functions of \( X_1 \) and \( Y_p \).
5. A priori estimates via the stability theorem

We consider the problem (1.1)-(1.5) assuming additionally:

\[ b > 0 \]
\[ h(x) \geq 0 \text{ for all } x \in (0, b), \]
\[ h \text{ is continuously differentiable in a vicinity of } x = b \]
\[ h' \text{ is locally Hölder continuous there} \]
\[ f = 0 \]
\[ q(x, t) \leq 0 \text{ for all } (x, t) \in \Omega_T, \]
\[ q \text{ is locally Hölder continuous in every bounded subset of } \Omega_T \]
\[ \mu(x, t) = \mu(t) \text{ for all } t \in [0, T]. \]

For a brief formulation of our problem we need the notion of the free boundary operator. To stay within the framework of classical solutions, we use the function space \( C_{pw}[0, T] \) of piecewise continuous functions in \([0, T]\).

5.1 Definition: Regard \((b, h, q, \lambda, \mu)\) as fixed. The free boundary operator is the following one:

\[ S: \{ g \in C_{pw}[0, T] | g(t) \leq 0 \text{ for } 0 \leq t \leq T \} \rightarrow C[0, T] \]

\[ g \mapsto s \]

for some \( u \) \((u, s)\) solve (1.1)-(1.5) for the data \((b, h, g, g, 0, \lambda, \mu)\).

The problem of interest is not the real inverse Stefan problem i.e. the reconstruction of the flux data \( g^* \) for a given boundary \( s^* \), but rather a question what we prefer to call

Problem: Steering to a boundary

\( s^* \) given (= the desired boundary)

Interest: Realization of \( s^* \) as a free boundary i.e. as \( Sg \) for some \( g \)

Approximate solution: \( \bar{g} \) such that \( Sg \) "near by" \( s^* \)

In the following we assume that \( s^* \) is in fact a free boundary with same additional regularity, i.e. \( s^* \) is continuously differentiable in \([0, T]\) and there is a \( g^* \in C_{pw}[0, T], g^*(t) \leq 0 \) for \( 0 \leq t \leq T \) such that

\[ Sg^* = s^*. \]

The operator \( F \) we define next is opposite to \( S \) affine-linear and will play a major role in the approximation technique, we will propose for the problem.

5.2 Definition: Regard \((b, h, q, \lambda, \mu)\) as fixed.

For given \( g \in C_{pw}[0, T] \) \( \bar{u} \) solves

\[ L\bar{u} = q \text{ in } D_T(s^*) \]

\[ \bar{u}(x, 0) = h(x) \text{ for } 0 \leq x \leq b \]

\[ \bar{u}_x(0, t) = g(t) \text{ for } 0 < t < T \]

\[ \bar{u}(s^*(t), t) = 0 \]

then the flux operator \( F \) is defined by

\[ Fg = \bar{u}_x(s^*(\cdot), \cdot) \]

Let \( r^* = \lambda(s^*(\cdot), \cdot)s^* + \mu \) then

\[ (5.2) \quad Sg^* = s^* \text{ is equivalent to } Fg^* = r^* \]

Furthermore we have from the stability theorem 3.1 the following

5.3 Corollary: There is a constant \( C \geq 0 \) such that for all
\[ g \in C_{pw}[0,T] \text{ with } g(t) \leq 0 \text{ for } 0 \leq t \leq T \]

\[ |(Sg)(t) - s^*(t)| \leq C \int_0^t |(Fg)(\tau) - r^*(\tau)| d\tau \text{ for } t \in [0,T] \]

**Proof:** \( Sg \) is with some \( u \) according to the definition a solution of (1.1)-(1.5) for the data \((b, h, g, q, 0, \lambda, \mu)\). On the other hand, let \( \tilde{u} \) be the solution of (5.1), then

\[ \tilde{u}_x(s^*(\cdot), \cdot) = Fg = \lambda (s^*(\cdot), \cdot) s^* + \mu + Fg - r^*. \]

Therefore \((\tilde{u}, s^*)\) solve (1.1)-(1.5) for the data \((b, h, g, q, 0, \lambda, \mu)\) using \( \tilde{u} = u + Fg - r^* \). Now we can apply Theorem 3.1 leading to the assertion.

This relation advises to handle the defect of \( Fg^* = r^* \), but the defect of \( Sg^* = s^* \) in a numerical scheme. One possibility to realize this idea is:

**Numerical scheme:** Let \( X_k \) be the space of piecewise constants on \([0,T]\) for a fixed mesh with meshsize \( k \), \( p \in \{1, \infty\} \).

Minimize \( \|Fg - r^*\|_{L^p[0,T]} \) with respect to \( g \in X_k \) \( g(t) \leq 0 \) for all \( 0 \leq t \leq T \).

This is a finite dimensional optimization problem with linear inequality constraints, which is quadratic for \( p = 2 \) and linear for \( p = 1, \infty \). For \( p \in \{1, \infty\} \) the unique existence and for \( p \in \{1, \infty\} \) at least the existence of an approximate solution \( g_k \) is clear.

We end this section by an order of convergence estimate based on Corollary 5.3, which shows that \( Sg_k \) has a higher order of convergence than provided by the approximation power of \( X_k \).

**5.4 Theorem:** Assume that \( g^* \in H^1[0,T] \). Let \( g_k \) be an approximate solution. Then for \( t \in [0,T] \):

\[ |(Sg_k)(t) - s^*(t)| \leq C \|g^*\|_{H^1[0,T]} k^2 \]

with a constant \( C > 0 \) independent of \( k \).

**Proof:** \( C \) will denote different constants at different places.

Let \( P_k : L^2[0,T] \to X_k \) be the orthogonal projection, then

\[ \|P_k \psi - \psi\|_{H^{-1}[0,T]} \leq C \|\psi\|_{H^1[0,T]} k^2 \]

(5.4) \( \psi(t) \leq 0 \) for \( 0 \leq t \leq T \) \( \Rightarrow (P_k \psi)(t) \leq 0 \) for \( 0 \leq t \leq T \)

for all \( \psi \in H^1[0,T] \)

- \( H^{-1}[0,T] \) denotes the dual space of \( H^1[0,T] \) - \( t \in [0,T] \).

Corollary 5.3 implies

\[ |(Sg_k)(t) - s^*(t)| \leq C \|Fg_k - r^*\|_{L^1[0,T]} \]

\[ \leq C \|Fg_k - r^*\|_{L^p[0,T]} \]

\[ \leq C \|F_k g^* - r^*\|_{L^p[0,T]} \]

because of \( P_k g^* \leq 0 \) (from (5.4))

\[ \leq C \|P_k g^* - Fg^*\| \]

\[ \leq C \|P_k g^* - g^*\|_{H^{-1}[0,T]} \]

This last estimate is a consequence of the extreme smoothing of \( F : \)

\( F : C[0,T], \|\cdot\|_{H^{-n}[0,T]} \to C[0,T] \) is continuous for all \( n \in \mathbb{N} \). (Compare Knabner [7]).

Finally using (5.3) we get the assertion.


DIFFERENTIABILITY OF THE FREE-BOUNDARY OPERATOR
FOR A TWO-PHASE STEFAN PROBLEM

H.-J. Kornstaedt
Freie Universität Berlin
III. Mathematisches Institut
H. Krüger
Naturwissenschaftliche Fakultät der Universität Augsburg
8900 Augsburg

Abstract: In this paper we shall deal with a class of two-phase Stefan-type problems for the heat equation in one space dimension. As our main result, we shall prove, that the solution operator \( S \) which, to each tuple of initial and boundary data, assigns the corresponding free boundary, is, under appropriate assumptions, arbitrary often Fréchet differentiable.

1. Introduction

We consider the following one-dimensional two-phase Stefan problem

\[
\begin{align*}
\alpha_1 u_{1xx}(x,t) - u_1(x,t) &= 0, \\
(x,t) &\in \Omega_1(s), \\
\quad u_1(x,0) &= h_1(x), \\
\quad c_1 < x < 0, \\
\quad u_{1x}(c_1,t) &= \rho_1(u_1(c_1,t),f_1(t),t), \\
\quad 0 < t \leq T, \\
\quad u_1(s(t),t) &= 0, \\
\quad 0 < t \leq T,
\end{align*}
\]