Bound-preserving flux limiting schemes for DG discretizations of conservation laws with applications to the Cahn–Hilliard equation

von

F. Frank, A. Rupp & D. Kuzmin

No. 401 2019
Bound-preserving flux limiting schemes for DG discretizations of conservation laws with applications to the Cahn–Hilliard equation

Florian Frank, Andreas Rupp, Dmitri Kuzmin

Friedrich-Alexander-Universität Erlangen-Nürnberg, Department Mathematik, Cauerstraße 11, 91058 Erlangen, Germany
Technische Universität Dortmund, Fakultät für Mathematik, Vogelpothsweg 87, 44227 Dortmund, Germany

Abstract

Many mathematical models of computational fluid dynamics involve transport of conserved quantities which must lie in a certain range to be physically meaningful. The analytical or numerical solution \( u \) of a scalar conservation law is said to satisfy a maximum principle (MP) if global bounds \( u_* \) and \( u^* \) exist such that \( u_* \leq u \leq u^* \) holds in the domain of definition. To enforce such inequality constraints at least for element averages in the context of discontinuous Galerkin (DG) methods, the numerical fluxes must be defined and constrained in an appropriate manner. In this work, we introduce a general framework for calculating fluxes that produce non-oscillatory DG approximations and preserve relevant global bounds for element averages even if the exact solution of the PDE violates them due to modeling errors. The proposed methodology is based on a combination of flux and slope limiting. The (optional) slope limiter adjusts the gradients to impose local bounds on pointwise values of the high-order DG solution which is used to calculate the fluxes. The flux limiter constrains changes of element averages so as to prevent violations of global bounds. Since manipulations of the target flux may introduce a consistency error, it is essential to guarantee that physically admissible fluxes remain unchanged. We propose two kinds of flux limiters which meet this requirement. The first one is of monolithic type and its time-implicit version requires the iterative solution of a nonlinear problem. Only a fully converged solution is provably bound-preserving. The explicit version of this limiter is subject to a time step restriction which we derive in this article. The second limiter is an iterative version of the multidimensional flux-corrected transport (FCT) algorithm. Our fractional step limiting (FSL) approach guarantees the MP property for each iterate but avoidable consistency errors may occur if the iterative process is terminated too early. Practical applicability of the proposed iterative limiters is demonstrated by numerical studies for the advection equation (hyperbolic, linear) and the Cahn–Hilliard equation (parabolic, nonlinear).

Keywords: Discontinuous Galerkin method, maximum principle, phase-field equation, slope limiting, flux limiting, flux corrected transport.

1. Introduction and motivation

Maximum principles play a vitally important role in the development of bound-preserving numerical methods for conservation laws. In many applications of practical interest, a physically admissible approximation to the conserved quantity must be bounded above and/or below. For example, volume or mass fractions of solute chemical species or disperse phases transported by a solvent should be conserved and stay within the range of \([0, 1]\). However, even exact weak solutions of the corresponding conservation laws may violates the physical bounds if the velocity of the carrier fluid is not exactly divergence-free [1], e. g., since it is calculated by solving the incompressible Navier–Stokes equations numerically or when a compressible flow model is considered. If the exact solution is provably bound-preserving, numerical approximations can be forced to satisfy the corresponding inequality constraints by using artificial diffusion operators or limiters to control fluxes, slopes, cell/element averages, and/or pointwise solution values.

The main focus of this work is on the construction of a flux-limiting technique for high-order discontinuous Galerkin (DG) finite element approximations enabling the optional use of subsequent slope limiting. Following the design philosophy of algebraic flux correction schemes [2, 3], we enforce inequality constraints for element averages by adjusting numerical fluxes into/from troubled elements. For this purpose, we first propose a monolithic limiter...
which introduces flux-correction factors nonlinearly depending on the discrete solution. Based on this, we propose a second “fractional-step” limiter that exploits a predictor-corrector strategy based on a generalization of Zalesak’s multidimensional flux corrected transport (FCT) algorithm [4]. While the monolithic limiter (ML) is promising for theoretical considerations where \( L^\infty \) bounds are required for the discrete unknown, the fractional step limiter (FSL) is preferred for practical simulations.

Similarly to the FCT scheme developed in [1] on the basis of a continuous Galerkin discretization, the new DG limiters may be configured to enforce global bounds even if the exact solution of the problem at hand violates them. To rule out formation of spurious undershoots/overshoots, our method cancels the offending fluxes or adjusts their magnitude in an iterative way until each element average becomes bounded as desired and the magnitude of rejected fluxes is minimized subjected to the given constraints. At the beginning of each time step, we correct the gradients of the DG solution using the vertex-based limiter presented in [5, 6]. This slope limiting step produces an approximate solution whose values at the vertices of the mesh are bounded in terms of physically admissible element averages in surrounding elements. A combination of flux and slope limiting makes it possible to constrain DG solutions without losing consistency if a bound-preserving consistent approximation exists. Consistency errors are introduced only if this is necessary to compensate a drawback of the mathematical model or of the given data. Numerical studies of such physics-conforming limiting approaches are conducted for piecewise-linear DG discretizations of the linear advection equation and of the (nonlinear) Cahn–Hilliard equation with different types of potentials.

### 1.1. Maximum principles in conservation laws

Let \( J := (0, T] \) denote a time interval with end time \( T > 0 \) and let \( \Omega \subset \mathbb{R}^d \), \( d \in \mathbb{N} \) be a polygonally/polyhedrally bounded domain with Lipschitz boundary \( \partial \Omega \). We consider a general form of the conservation equation

\[
\begin{align*}
\partial_t u + \nabla \cdot j(u, \nabla u, \Delta u, \ldots) &= 0 \quad \text{in } J \times \Omega, \\
\quad u &= u^0 \quad \text{on } [0] \times \Omega
\end{align*}
\]

for an unknown \( u : J \times \overline{\Omega} \to \mathbb{R} \) and given initial data \( u^0 : \Omega \to \mathbb{R} \). In addition, a closed-form expression of the vector-valued flux \( j : J \times \overline{\Omega} \to \mathbb{R}^d \) possibly nonlinearly depending on \( u \) and its derivatives and appropriate boundary conditions shall be given. The form of the flux \( j \), the type of boundary conditions, and the physical interpretation of \( u \) depend on the considered physical process, cf. Example 1 below.

Mathematical models obeying (1) often are physically not meaningful if the unknown \( u \) exceeds certain bounds, leading e.g. to negative concentrations/densities or to a negative percentage or a percentage larger than one in the case of multi-component mixtures. This means, that \( u \) has to fulfill the maximum principle (MP)

\[ \forall (t, x) \in J \times \Omega, \quad u_* \leq u(t, x) \leq u^* \]  

for certain global bounds \( u_* < u^* \) with \( u_*, u^* \in \mathbb{R} \). In particular, the initial data should satisfy \( u^0(x) \in [u_*, u^*] \). The bounds \( u_* = -\infty \) and \( u^* = \infty \) are allowed, but in these cases the corresponding inequality in (2) becomes strict and the initial and boundary data must be bounded away from \( \pm \infty \).

**Example 1** (Physical problems).

1. **Advection transport along a velocity field** \( v : \overline{\Omega} \to \mathbb{R}^d \) is modeled by supplementing (1) with

\[
\begin{align*}
\quad j &= v \cdot u \quad \text{in } J \times \Omega, \\
\quad u &= u^{in} \quad \text{on } J \times \partial \Omega^{in},
\end{align*}
\]

where \( u^{in} : \partial \Omega^{in} \to [u_*, u^*] \) describes the Dirichlet data at the inflow boundary \( \partial \Omega^{in} := \{ x \in \partial \Omega; \nu \cdot n < 0 \} \). Solutions of \( \{ (1), (3) \} \) satisfy the MP (2) if \( v \) is solenoidal, i.e., \( \nabla \cdot v = 0 \) [3]. Without the divergence-free assumption, the MP does not hold in general [1].

2. **Diffusive transport of a dissolved chemical species in a solvent occupying a domain \( \Omega \) with impermeable boundary \( \partial \Omega \) is described on an averaged spatial scale by Fick’s law yielding**

\[ j := -K \nabla u \quad \text{in } J \times \Omega, \]

[2]
\[ j \cdot n = 0 \quad \text{on } J \times \partial \Omega \] (4b)

in conjunction to (1), with \( K > 0 \). The unknown \( u \) typically represents a concentration allowed to attain nonnegative values bounded away from \(+\infty\). Solutions of \{ (1),(4) \} satisfy the MP (2).

3. Phase separation of an incompressible, immiscible binary mixture at a constant temperature, which is the alignment of the mixture into spatial domains predominated by one of the two fluids, is described by the Cahn–Hilliard equation, where \( u \) is a dimensionless order parameter representing the difference in the mass fractions of the two components:

\[
\begin{align*}
  j & := -M(u) \nabla \phi & \text{in } J \times \Omega, \\
  \phi & := \Psi'(u) - \kappa \Delta u & \text{in } J \times \Omega, \\
  \nabla u \cdot n & = 0 & \text{on } J \times \partial \Omega, \\
  j \cdot n & = 0 & \text{on } J \times \partial \Omega,
\end{align*}
\]

where the parameter \( \kappa > 0 \) controls the interface width and \( M \geq 0 \) limits the flux \( j \) in a possibly nonlinear fashion. The nonlinear potential \( \Psi \) attains local minima at or close to \([-1, 1]\). Solutions \( u \) of \{ (1),(5) \} are physically meaningful only in the interval \([-1, 1]\). Whether or not \{ (1),(5) \} satisfies the MP (2) with \( u_c = -1, u^* = 1 \) depends on the choice of mobility \( M \) and of potential \( \Psi \) [7, 8]. Further model details are given in Section 6.1.

4. Other scalar conservation laws obeying (1) include the Burgers equation, the Nernst–Planck equation, the Buckley–Leverett equation, and the thin-film equation.

1.2. Discontinuous Galerkin methods

Discontinuous Galerkin (DG) methods are derived using polynomial test functions of various order with local (i.e. elementwise) support and the definition of numerical fluxes across element boundaries, which provides great flexibility. As such, they can be interpreted as a generalization of finite volume methods overcoming the restriction to locally constant approximations, lesser numerical/artificial diffusion, and enabling a wider range of grid geometries. One major advantage of DG is that basic conservation laws (e.g. local mass conservation) are straightforward to incorporate into the numerical scheme by having control over the numerical fluxes and the locality of the method. Basis functions may be chosen of nodal or modal type, each implying certain advantages [9, 10]. Moreover, DG can easily deal with hanging nodes and is independent of the local polynomial order, thus naturally leading to hp-adaptation [11]. Independent of the approximation order, standard DG schemes yield a stencil with one-vicinity across element faces which is especially attractive in parallel computing. The major drawback of DG is a possibly much larger number of global degrees of freedom compared to finite element discretizations of the same order (cf. [12, Sec. 2.12]), which is in particular important for implicit methods. For a historical outline of DG methods, see [13].

1.3. Limiting schemes for scalar conservation laws

In high-order DG methods for conservation laws, problem-dependent discrete maximum principles (DMP, e.g., positivity preservation and boundedness by the maxima/minima of the data in neighbor elements) are commonly enforced using slope limiting [14, 15, 16, 17, 18, 5, 6] to adjust the derivatives of the numerical solution. Detailed analysis of slope limiters satisfying local maximum principles on unstructured triangular meshes was recently performed in [18]. Numerical fluxes defined in terms of slope-limited solutions may require further adjustment. If element averages are found to violate DMPs, they can be fixed using a flux limiter [19]. As shown by Zhang and Shu [20], the imposition of global bounds on element averages does not necessarily degrade the rates of convergence to MP-satisfying smooth solutions. The most recent trends in the design of accuracy-preserving limiters for DG discretizations of hyperbolic systems include the use of \textit{a posteriori} subcell limiters based on customized combinations of physical and numerical admissibility criteria [21, 22].

In this paper, we introduce flux limiters designed to enforce the MP (2) with global bounds for element averages. If a pointwise global MP is desired, the target fluxes are calculated using slope-limited data, which rarely produce nonphysical mean values. To minimize the levels of numerical diffusion and possible consistency errors, a good flux limiting procedure should recognize harmless fluxes and leave them unchanged. Limiters satisfying this requirement
can be developed using optimization-based approaches [23] but the associated computational cost is exorbitant. In this work, we avoid solution of global optimization problems by using iterative algorithms in the FSL which produce quasi-optimal constrained fluxes.

2. DG formulation for scalar conservation equations

2.1. Weak formulation

Formally multiplying (1) by a smooth test function \( w : E \rightarrow \mathbb{R} \), integrating over the closed Lipschitz subset \( E \) of \( \overline{\Omega} \), and integration by parts yields the weak formulation

\[
\int_E w \partial_t u(t) - \int_E \nabla w \cdot j(t) + \int_{\partial E} w j(t) \cdot n_E = 0
\]

for almost every \( t \in J \), where \( n_E \) denotes the outward unit normal on \( \partial E \). The flux \( j \) is a function of \( u \) and its derivatives, which we suppress in the notation.

2.2. Discrete setting

Henceforth, let \( \mathcal{E}_h := \{ E \} \) be a non-overlapping partition of \( \overline{\Omega} \subset \mathbb{R}^d \) by closed, non-degenerated, and convex “elements” \( E \) with Lipschitz boundaries and a maximum diameter of \( h \), i.e., \( \overline{\Omega} = \bigcup_{E \in \mathcal{E}_h} E \). Let \( \Gamma_h \) denote the union of interior faces, i.e., faces connecting two elements, cf. Figure 1a. Let \( P_p(E) \) denote a finite dimensional polynomial space on \( E \) with polynomial order at most \( p \). An element may have an arbitrary number of faces as we do not require a nodal/Lagrange property. The broken polynomial space on the triangulation \( \mathcal{E}_h \) is given as

\[
P_p(\mathcal{E}_h) := \prod_{E \in \mathcal{E}_h} P_p(E).
\]

Elements of \( P_p(\mathcal{E}_h) \) are in general discontinuous across \( \Gamma_h \).

For each interior face \( \Gamma_h \supset (\partial E \cap \partial E') := e_{E,E'} \), let the average \( \langle \cdot \rangle \) and the jump \( \llbracket \cdot \rrbracket \) of a scalar quantity \( w \) on face \( e_{E,E'} \) be defined by

\[
\langle w \rangle := \frac{1}{2} w|_E + \frac{1}{2} w|_{E'} \quad \text{and} \quad \llbracket w \rrbracket := w|_E n_E + w|_{E'} n_{E'}.
\]

The average of a vector-valued quantity is defined component-wise. Note that the jump of a scalar is a vector and that the definition is invariant regarding swapping \( E \) and \( E' \).

\( H_{E,E'}(1) \) is the numerical flux from \( E \) to \( E' \) across face \( e_{E,E'} = \partial E \cap \partial E' \), i.e., in direction of the outward unit normal \( n_E \). Clearly, \( n_E = -n_{E'} \) and local mass conservation requires \( H_{E,E'} = -H_{E',E} \). \( \Gamma_h \) is the union of \( e_{E,E'}, e_{E',E''}, \) and \( e_{E''E'} \).

(b) The bounds \( u_{E,1}^{E'}, u_{E,1}^{E} \) defined in (21) are determined from integral means \( u_{0,E'} \) of elements \( E' \) within the neighborhood of the vertex \( a_{E,1} \in E \) (filled area).

Figure 1: Admissible grid with 7 elements. Boundaries may additionally be curved.
2.3. Space-discrete problem and numerical fluxes

A general DG formulation of (6) on an interior element \( E \in \mathcal{E}_h \) with \( \partial E \subset \Gamma_h \) then reads:

Find \( u_h(t) \in \mathbb{P}_p(\mathcal{E}_h) \) such that for a.e. \( t \in J \),

\[
\forall \, w_h \in \mathbb{P}_p(E), \quad \int_E w_h \partial_t u_h(t) - \int_E \nabla w_h \cdot j_h(t) + \sum_{E \neq E'} \int_{\partial E \cap \partial E'} H_{E,E'}(t)(w_h) = 0 \tag{7}
\]

holds. By the notation \( H_{E,E'}(w_h) \) we also implicitly include dependencies on spatial derivatives of \( w_h \) and \( u_h \). The form \( H_{E,E'}(1) \) represents the local numerical flux from \( E \) to \( E' \) across the face \( e_{E,E'} \), cf. Figure 1a, which depends on the considered problem (i.e., on the definition of \( j \) in (1)), on the chosen DG method, and possibly on geometric properties of \( E \). The flux \( H_{E,E'}(1) \) is supposed to approximate \( j(t) \cdot n_E \) on \( \partial E \) and affects the stability and accuracy of the numerical scheme as well as sparsity and symmetry of the resulting matrix.

For a boundary element \( E \in \mathcal{E}_h \) with \( \partial E \cap \partial \Omega \neq \emptyset \), depending on the type of boundary condition, the boundary integral is to be substituted by an appropriate flux and/or by prescribed boundary data.

Example 2 (Numerical fluxes).

For simplicity, let \( E \in \mathcal{E}_h \) in (7) be an interior element, i.e., \( \partial E \subset \Gamma_h \).

1. For the advection equation \((1),(3)\), full upwinding on element boundaries yields

\[
H_{E,E}(w) := w|_E u^1 \cdot n_E ,
\]

on \( e_{E,E'} \), where \( u^1 := u|_E \) if \( v \cdot n_E > 0 \) and \( u^1 := u|_{E'} \) otherwise [24].

2. For the diffusion equation \((1),(4)\), various interior penalty DG methods are given by

\[
H_{E,E'}(w) := -w|_E \triangleq \nabla u \cdot n_E + \frac{\varepsilon}{2} \nabla w|_E \cdot \| u \| + \frac{\sigma}{h} w|_E \| u \| \cdot n_E \\
= -\frac{1}{2} w|_E (\nabla u|_E + \nabla u|_{E'}) \cdot n_E + \frac{\varepsilon}{2} \nabla w|_E \cdot n_E (u|_E - u|_{E'}) + \frac{\sigma}{h} |w|_E (u|_E - u|_{E'}) \tag{9}
\]

on \( e_{E,E'} \). The choices \( \sigma > 0 \) and \( \varepsilon = -1, 0, 1 \) yield the symmetric interior penalty [25, 26], incomplete interior penalty [27], nonsymmetric interior penalty [28], respectively. The choice \( \sigma = 0, \varepsilon = 1 \) leads to the Oden–Babuška–Baumann method [29]. The choice of the denominator in \( \frac{\sigma}{h} \) is equivalent to other standard choices like \( |e|/\mathrm{diam} \) or \( |e|/\triangle \) if the mesh is assumed to be shape and contact regular in the sense of [30, Def. 1].

3. Consider the Cahn–Hilliard equation \((1),(5)\) with constant mobility \( M \). We set \( M := 1 \) since \( M \) represents a scaling in time \( t \). Application of an interior penalty DG on the mixed formulation yields

\[
H_{E,E'}(w) := -w|_E \langle \nabla \phi \rangle \cdot n_E + \frac{\varepsilon}{2} \nabla w|_E \cdot \| \phi \| + \frac{\sigma}{h} w|_E \| \phi \| \cdot n_E \tag{10}
\]

on \( e_{E,E'} \), where \( \phi(t) \in \mathbb{P}_p(\mathcal{E}_h) \) satisfies for all elements \( E \in \mathcal{E}_h \) the auxiliary problem (25b).

4. The proposed flux limiters can also be applied to other discontinuous Galerkin schemes such as the local discontinuous Galerkin method [32] or hybridizable discontinuous Galerkin schemes [33], where only the FSL preserves the efficiency advantage of hybrid methods (the ML would couple the local problems that have to be solved in a post-processing step).

In all above cases, \( H_{E,E} = -H_{E,E} \) holds guaranteeing that no “mass” is lost when transported across inter-element boundaries.

3. Flux limiter — enforcing a MP for local means

Classical flux limiters of FCT [4, 34] and TVD [35, 36] type use solution-dependent correction factors \( \alpha_{E,E'} \in [0, 1] \) to blend a low-order bound-preserving flux \( H_{E,E}' \) and a potentially troublesome high-order approximation \( H_{E,E} \).

In DG methods for hyperbolic conservation laws, \( H_{E,E}' \) is usually taken to be the Lax–Friedrichs flux based on
element averages. If no consistent flux approximation guarantees preservation of global bounds for the given PDE, then nonphysical solution behavior is caused by modeling errors, which can be corrected at the discrete level using \( \alpha_{E,E'} \) to limit the entire flux \( \mathcal{H}_{E,E'} \) rather than the flux difference \( \mathcal{H}_{E,E'}^{\text{high}} - \mathcal{H}_{E,E'}^{\text{low}} \). This idea goes back to the physics-based FCT algorithm which was designed in [1] to enforce the close-packing limit for volume fractions of a disperse phase in macroscopic models of dense suspension flows. In this section, we adapt it to the DG setting and propose new approaches for iterative limiting of \( \mathcal{H}_{E,E'} \). The objective is to minimize the magnitude of \( \sum_{E \neq E'} (1 - \alpha_{E,E'}) \mathcal{H}_{E,E'} \) for each \( E \in E_h \), i.e., to suppress as little flux as possible, subjected to the inequality constraints that guarantee the validity of physics-compatible DMPs for element averages.

### 3.1. Flux limiting and local mass conservation

The existence of a MP (2) for the solution of the continuous problem (1) is necessary but not sufficient for a discrete MP of the DG solution of (7), meaning that the DG solution may violate (2) in either case. In order to enforce a **discrete maximum principle for the local mean values**, the numerical flux in (7) must be adjusted whenever it would lead to mean values violating the bounds \( u_\ast, u' \). If the definition of \( \mathcal{H}_{E,E'} \) in terms of element averages (rather than one-sided limits of the high-order solution) produces a bound-preserving flux \( \mathcal{H}_{E,E'}^{\text{low}} \), then the problem can be solved using conventional flux or slope limiting. Otherwise, the nonphysical flux \( \mathcal{H}_{E,E'} \) needs to be limited using a suitably chosen correction factor \( \alpha_{E,E'} \in [0, 1] \). The flux-limited version of (7) reads

\[
\forall w_h \in \mathcal{P}_p(E), \quad \int_E w_h \partial_t u_h(t) - \int_E \nabla w_h \cdot \mathbf{j}_h(t) + \sum_{E \neq E'} \alpha_{E,E'} \int_{E \neq E'} \mathcal{H}_{E,E'}(w_h) = 0
\]

for a.e. \( t \in J \). The maximum admissible value of \( \alpha_{E,E'} \) depends on the local mean values of \( u_h \) on \( E \), i.e. \( \bar{u}_{h,E}(t) \), and \( E' \), i.e. \( \bar{u}_{h,E'}(t) \), and the net value of the numerical flux \( \mathcal{H}_{E,E'} \) with

\[
\mathcal{H}_{E,E'} := \int_{E \neq E'} \mathcal{H}_{E,E'}(1).
\]

The extreme cases are \( \alpha_{E,E'} = 1 \) meaning no limiting and \( \alpha_{E,E'} = 0 \) meaning that the flux between \( E \) and \( E' \) is suppressed completely. The limiting factor satisfies the **symmetry condition** \( \alpha_{E,E'} = \alpha_{E',E} \) ensuring the continuity of fluxes across inter-element boundaries, cf. Remark 1. It vanishes when the solution \( u \) reaches the bounds of the admissible range of \( [u_\ast, u'] \). The nonlinear character of the limiting coefficients produces a nonlinear problem—even if \( \mathcal{H}_{E,E'} \) is linear.

**Remark 1** (Local mass conservation).  
Consider an element \( E \in E_h \), for sake of a simple presentation, not adjacent to the boundary of \( \Omega \). From (1a), i.e. on the continuous level, it follows for the change of the local mean \( \bar{u}_E \) of \( u \) on \( E \) that

\[
\frac{d\bar{u}_E}{dt}(t) + \frac{1}{|E|} \int_{\partial E} \dot{u}_E \cdot n_E = 0
\]

for a.e. \( t \in J \). At semi-discrete level, choosing \( w_h|_E = 1 \) in (7) yields

\[
\frac{d\bar{u}_{h,E}}{dt}(t) + \frac{1}{|E|} \sum_{E \neq E'} \mathcal{H}_{E,E'} = 0
\]

with flux-limited analogue

\[
\frac{d\bar{u}_{h,E}}{dt}(t) + \frac{1}{|E|} \sum_{E \neq E'} \alpha_{E,E'} \mathcal{H}_{E,E'} = 0,
\]

where all fluxes are scaled by factors in the range \([0, 1]\). In DG discretizations of conservation laws, a positive flux into \( E \) must be balanced by a negative flux into \( E' \) and vice versa. This is obviously the case for \( \mathcal{H}_{E,E'} = -\mathcal{H}_{E,E'} \) and \( \alpha_{E,E'} = \alpha_{E,E'} \). Clearly, the use of \( \alpha_{E,E'} < 1 \) introduces a consistency error which should be kept as small as possible. Therefore, the properties of the target fluxes \( \mathcal{H}_{E,E'} \) and algorithms for practical calculation of \( \alpha_{E,E'} \) have a stronger
impact on the outcome of the limiting procedure than in FCT-like nonlinear schemes that use convex combinations of consistent flux approximations. The multiplication of $\mathcal{H}_{E,E'}$ by $\alpha_{E,E'} < 1$ reduces the rate of transport across the common boundary of elements $E$ and $E'$. This adjustment may be the only way to prevent undershoots/overshoots but aggressive limiting should be avoided because it may result in a nonphysical deceleration of transport processes and significantly increase the (local) time scale if inordinately small values of $\alpha_{E,E'}$ are employed.

Let $0 =: t_0 < t_1 < \ldots < t_{n_u} := T$ be a decomposition of $J$ into $N_u$ subintervals. After discretization in time by a time-implicit or time-explicit scheme that approximates the time derivative by a first-order difference quotient, (11) reads: Given $u^0 : \Omega \to [u_-, u^*]$, for $n \in \{1, \ldots, N_u\}$, find $u^n_h \in \mathbb{P}_p(\mathcal{E}_h)$ such that

$$\forall \, w_h \in \mathbb{P}_p(E), \quad \int_E w_h \frac{u^n_h - u^{n-1}_h}{\tau_n} - \int_E \nabla w_h : \tilde{f}_h^n + \sum_{E_{\alpha} \in E \neq E} \alpha_{E,E'}^\ell \int_{E_{\alpha} \cap E} H_{E,E'}^\ell (w_h) = 0 \quad (15)$$

for $E \in \mathcal{E}_h$, where $\tau_n := t_n - t_{n-1}$ denotes the time step size and $\ell \in \{n-1, n\}$. We distinguish here the time-explicit case ($\ell = n-1$, forward Euler method) from time-implicit cases $\ell = n$ (implicit $\theta$-scheme or IMEX methods). In implicit cases, $\tilde{f}_h^n, \alpha_{E,E'}^\ell$ and $H_{E,E'}^\ell (w_h)$ may nonlinearly depend on $u^{n-1}_h$ and $u^n_h$. Choosing $w_h = 1$ in (15) yields

$$\bar{u}^{n-1}_h = \bar{u}^n_h - \frac{\tau_n}{|E|} \sum_{E_{\alpha} \in E \neq E} \alpha_{E,E'}^\ell \mathcal{H}_{E,E'}^\ell . \quad (16)$$

The solution at time level zero is given by $\forall w_h \in \mathbb{P}_p(E), \ (w_h, u^0_h) \in \mathbb{P}_p(\mathcal{E}_h)$ for $E \in \mathcal{E}_h$.

A time step then consists of solving the flux-limited problem (15), where an additional closed formulation for $\alpha_{E,E'}^\ell$ must be given, cf. Section 3.2, or of the solution of (15) with all $\alpha_{E,E'}^\ell$ set to one, (i.e. no limiting) and a subsequent iterative correction of the numerical fluxes, cf. Section 3.3. In the first case, the limiting factors $\alpha_{E,E'}^\ell$ depend nonlinearly on the solution and may be treated implicit in time introducing a (possibly additional) nonlinearity to the problem or explicit in time implying a CFL condition.

3.2. Flux limiter 1 (monolithic limiter, ML)

Consider (15) at time level $n$ either in time-explicit ($\ell = n-1$) or time-implicit ($\ell = n$) version. Let

$$\alpha_{E,E'}^\ell := \begin{cases} \min(\alpha_{E,E'}^{-}, \alpha_{E,E'}^{+}) & \text{if } \mathcal{H}_{E,E'}^\ell < 0 \quad \text{(positive flux from } E' \text{ to } E), \\ \min(\alpha_{E,E'}^{+}, \alpha_{E,E'}^{-}) & \text{if } \mathcal{H}_{E,E'}^\ell > 0 \quad \text{(positive flux from } E \text{ to } E') \end{cases}$$

with net numerical fluxes $\mathcal{H}_{E,E'}^\ell$, acc. to (12), where

$$\alpha_{E,E'}^{+} := \min \left\{ 1, \gamma_n |E| \max \left\{ 0, u^* - \bar{u}_E^n \right\} \right\}, \quad \alpha_{E,E'}^{-} := \min \left\{ 1, \gamma_n |E| \max \left\{ 0, -\bar{u}_E^n \right\} \right\}. \quad (17)$$

In (17), $0 < \delta \ll 1$ is a small regularization parameter preventing division by zero. The parameter $\gamma_n \gg 0$ controls the amount of flux-limiting and should be chosen as large as possible: As $\gamma_n$ increases, the correction factors $\alpha_{E,E'}^{\pm}$ approach unity, unless the mean value $\bar{u}_E^n$ has already reached the global upper/lower bound and no positive/negative fluxes can be accepted anymore. Hence, larger values of $\gamma_n$ imply smaller consistency errors but the time step restriction for explicit schemes becomes more severe, cf. (18). In the time-implicit case, an increase in the value of $\gamma_n$ has an adverse effect on the convergence behavior of nonlinear solvers. Therefore, an implicit upper bound is imposed on $\gamma_n$ by fixed-point theorems that guarantee convergence.

Theorem 1 (Maximum principle for local means).
For $\ell \in \{n-1, n\}$, let $u^n_h, n \in \{1, \ldots, N_u\}$ be the solution of (15) and let the initial data satisfy $\forall E \in \mathcal{E}_h, \, \bar{u}_E^0 \in [u_-, u^*]$. 7
For $\ell = n - 1$ (time-explicit case), let additionally

$$\gamma_n \leq \frac{1}{2\tau_n}$$

hold. Then, for $n \in \{1, \ldots, N_{sl}\}$

$$\forall E \in E_0, \quad \overline{u}_E^n \in [u_*, u^*].$$

**Proof.** Consider (15). Assume that $u_* < \overline{u}_E^{n-1} < u^*$ (the proofs for the cases $\overline{u}_E^{n-1} = u_*$ and $\overline{u}_E^{n-1} = u^*$ are obvious).

$$\overline{u}_E^n = \overline{u}_E^{n-1} - \frac{\tau_n}{|E|} \sum_{E \neq E'} \alpha_{E,E'}^\ell \mathcal{H}_{E,E'}^\ell$$

$$= \overline{u}_E^{n-1} + \frac{\tau_n}{|E|} \sum_{E \neq E'} \alpha_{E,E'}^\ell \left( \max \left\{0, -\mathcal{H}_{E,E'}^\ell \right\} - \max \left\{0, \mathcal{H}_{E,E'}^\ell \right\} \right)$$

$$= \overline{u}_E^{n-1} + \tau_n \left( \sum_{E \neq E'} \alpha_{E,E'}^\ell \max \left\{0, -\mathcal{H}_{E,E'}^\ell \right\} \frac{u^* - \overline{u}_E^{n-1}}{u^* - \overline{u}_E^{n-1}} \right) + \tau_n \left( \sum_{E \neq E'} \alpha_{E,E'}^\ell \max \left\{0, \mathcal{H}_{E,E'}^\ell \right\} \frac{\overline{u}_E^{n-1} - u_*}{\overline{u}_E^{n-1} - u_*} \right)$$

$$= \overline{u}_E^{n-1} + \tau_n \left( \gamma_E^+ (u^* - \overline{u}_E^{n-1}) + \gamma_E^- (u_* - \overline{u}_E^{n-1}) \right),$$

where the time index $\ell$ in $\gamma_E^{\ell \pm}$ is suppressed. For $\ell = n$, it follows

$$\left(1 + \tau_n (\gamma_E^+ + \gamma_E^-)\right) \overline{u}_E^n = \overline{u}_E^{n-1} + \tau_n \gamma_E^+ u^* + \tau_n \gamma_E^- u_*.$$

Since all factors are non-negative, $\overline{u}_E^n$ is a convex combination of $\overline{u}_E^{n-1}$, $u^*$, and $u_*$. For $\ell = n - 1$, it follows

$$\overline{u}_E^n = (1 - \tau_n (\gamma_E^+ + \gamma_E^-)) \overline{u}_E^{n-1} + \tau_n \gamma_E^+ u^* + \tau_n \gamma_E^- u_*.$$

Since $\gamma_E^+ \geq 0$, it remains to show that $\tau_n (\gamma_E^+ + \gamma_E^-) \leq 1$. In the definition of $\gamma_E^-$, we only need to consider the case $\mathcal{H}_{E,E'}^{n-1} > 0$, otherwise, the summand is zero. Hence, using the corresponding definition of $\alpha_{E,E'}^{n-1}$, it follows

$$\gamma_E^- = \sum_{E \neq E'} \min \left\{ \alpha_{E,E'}^{n-1,+}, \alpha_{E,E'}^{n-1,-} \right\} \max \left\{0, \frac{1}{|E|} \mathcal{H}_{E,E'}^{n-1} \right\} \leq \frac{\alpha_{E,E'}^{n-1,+} - \alpha_{E,E'}^{n-1,-}}{\overline{u}_E^{n-1} - u_*} \sum_{E \neq E'} \max \left\{0, \frac{1}{|E|} \mathcal{H}_{E,E'}^{n-1} \right\} \overline{u}_E^{n-1} - u_* \tau_n \gamma_E^- u_* + \tau_n \gamma_E^- u_*.$$
\[
\gamma \max \left\{ 0, \pi_E^{n-1} - u_*, \right\} \sum_{E' \in E} \max \left\{ 0, \frac{1}{|E'|} \mathcal{H}_{E', E}^{n-1} \right\} \leq \gamma_n .
\]

Analogously, one can show \( \gamma_E^* \leq \gamma \). Hence, \( \pi_E^n \) is a convex combination of \( \pi_E^{n-1} \), \( u_* \), and \( u_* \) since \( \tau_n \) satisfies (18).

For the time-implicit monolithic limiter, the parameter \( \gamma_n \) is to be chosen as large as possible such that the nonlinear solver does not diverge or is not converging too slowly. Since intermediate solutions are not guaranteed to be bound-preserving, it is essential to obtain a fully converged solution at the final flux correction cycle. To that end, \( \gamma_n \) may be adjusted in each time step depending on the number of nonlinear iterations. If Newton’s method is chosen as nonlinear solver, it is difficult to assemble the Jacobian, as all \( \alpha_{E', E}^n \) depend on the implicit solution. To circumvent this, Jacobian-free methods may be used such as the Jacobian-free Newton–Krylov method [37, 38]. For the time-explicit case, the largest possible choice of \( \gamma_n \) is 1/(2\( \tau_n \)), cf (18), which is the optimal choice.

### 3.3. Flux limiter 2 (fractional step limiter, FSL)

We follow the idea of the overshoot limiter presented in [1]. Instead of limiting target fluxes that depend on the unknown solution and that must be updated step-by-step, we calculate the solution \( u_{h, E}^{\text{high}} \) using unlimited fluxes (i.e., \( \alpha_{E, E'} \equiv 1 \)). Then we decompose the difference between \( u_{h, E}^{\text{high}} \) and a bound-preserving “backup solution” \( u_{h, E}^{\text{low}} \) into “phoenical” fluxes \( \mathcal{H}_{E, E'}^{\text{high}} = \mathcal{H}_{E, E'}^{\text{high}} - \mathcal{H}_{E, E'}^{\text{low}} \), which can be used to reconstruct \( u_{h, E}^{\text{high}} \) from \( u_{h, E}^{\text{low}} \) “like a phoenix”. When it comes to flux limiting, we multiply \( \mathcal{H}_{E, E'}^{\text{high}} \) by suitably defined correction factors \( \alpha_{E, E'} \) and add the result to \( u_{h, E}^{\text{low}} \). This limiting strategy is based on Zalesak’s FCT algorithm [4], and we use his formula to calculate \( \alpha_{E, E'} \) but perform multiple correction cycles to recover the target flux to the greatest extent possible. In contrast to the above monolithic limiter, all intermediate solutions are guaranteed to be bound-preserving. If the process of iterative flux correction begins with \( u_{h, E}^{\text{low}} = u_{h, E}^{n-1} \), however, a consistency error is introduced and may be quite significant if the iteration process is terminated at early stages (see below for detailed explanations and numerical examples).

**Algorithm 1** (Fractional-step limiter with global bounds for mean values).

Let \( n \in \{1, \ldots, N_u\} \) be a fixed time level and \( u_{h, E}^{n-1} \) be given.

1. Solve non-limited problem: Solve (15) with \( \alpha_{E, E'} = 1 \) for the high-order solution \( u_{h, E}^{\text{high}} \in \mathbb{P}_p(\mathcal{E}_h) \).
   
   Remark: For the mean values, it holds for \( E \in \mathcal{E}_h \) (cf. (14))
   \[
   u_{h, E}^{\text{high}} = u_{h, E}^{n-1} - \frac{\tau}{|E|} \sum_{E_i \in E \neq E} \mathcal{H}_{E_i, E}^{\text{high}} .
   \]

2. Select low order solution: Construct an approximation \( u_{h, E}^{\text{low}} \in \mathbb{P}_p(\mathcal{E}_h) \) of (15) that satisfies the MP (2) for local means.
   
   Remark: For the mean values, it holds for \( E \in \mathcal{E}_h \) (cf. (14))
   \[
   u_{h, E}^{\text{low}} = u_{h, E}^{n-1} - \frac{\tau}{|E|} \sum_{E_i \in E \neq E} \mathcal{H}_{E_i, E}^{\text{low}} .
   \]

   If the continuous problem (1a) satisfies a MP, a low-order solution may be given by choosing zeroth order polynomials. In that case, the difference between high and low order fluxes is limited, cf. Step 3. Otherwise, \( u_{h, E}^{\text{low}} = u_{h, E}^{n-1} \) is a valid choice implying that \( \mathcal{H}_{E, E'}^{\text{low}} = 0 \). In this case the whole numerical flux is limited.

3. Initialize iteration: Choose tolerances \( 0 < \varepsilon_1, \varepsilon_2 < 1 \) and set
   \[
   m := 1 , \quad u_{h, E}^{(m-1)} := u_{h, E}^{\text{low}} , \quad \mathcal{H}^{(m-1)} := \mathcal{H}_{E, E}^{\text{high}} - \mathcal{H}_{E, E}^{\text{low}}
   \]
   for \( E, E' \in \mathcal{E}_h \).
   
   Remark: Steps 4–7 require the information of local means of \( u_{h, E}^{(m)} \) only, disregarding the higher-order parts.
4. Find limiting factors and compute limited averages: Seek \( \alpha_{E,E'}^{(m)} \in [0, 1] \) for \( E, E' \in \mathcal{E}_h \) with \( \partial E \cap \partial E' \neq \emptyset \) and \( E \neq E' \) such that
\[
\bar{u}^{(m)}_{h,E} = \bar{u}^{(m-1)}_{h,E} - \frac{\tau}{|E|} \sum_{E \cap E' \neq E} \alpha_{E,E'}^{(m)} \mathcal{H}^{(m-1)}_{E,E'}
\]
satisfies \( u_\ast \leq \bar{u}^{(m)}_{h,E} \leq u' \) (cf. (2)) for \( E \in \mathcal{E}_h \) using Algorithm 2. Define \( \bar{u}^{(m)}_{h,E} \) by this equation.

5. Recycle the suppressed fluxes:
\[
\mathcal{H}_{E,E'}^{(m)} := (1 - \alpha_{E,E'}^{(m)}) \mathcal{H}_{E,E'}^{(m-1)}.
\]

6. Repeat or terminate: If
\[
\max_{E,E' \in \mathcal{E}_h} |\mathcal{H}^{(m)}_{E,E'}| < \epsilon_1 \quad \text{or} \quad \max_{E,E' \in \mathcal{E}_h} |\mathcal{H}^{(m)}_{E,E'} - \mathcal{H}^{(m-1)}_{E,E'}| < \epsilon_2 \quad \text{(for } m \geq 2 \text{ only)}
\]
then go to Step 7. Otherwise, set \( m \leftarrow m + 1 \) and go to Step 4.

7. Reconstruct high-order mean-limited solution: For \( E \in \mathcal{E}_h \), set
\[
u_{h,E}^\ast := u_{h,E}^\ast = \nu_{h,E}^\text{high} = \bar{u}^{(m)}_{h,E}.
\]

Remark: The first two terms on the right represent the high-order part of \( u_{h}^\text{high} \) without the constant part.

Obviously, the solution of Algorithm 1 converges at least with the order of the convergence of the low order scheme since the solution is interpolated between the solution of the low-order scheme and the limited high-order solution in each time step. The number of iterations to be performed depends on the definitions of \( u_{h}^\text{low} \) and \( \alpha_{E,E'} \). The calculation of correction factors that minimize \( (1 - \alpha_{E,E'}) \mathcal{H}_{E,E'} \) subjected to DMPs requires the solution of a global inequality-constrained optimization problem, which is possible [23] but usually too expensive for practical purposes. Assuming the worst-case scenario in which all limited fluxes have the same sign and conspire to create an undershoot/overshoot, a closed-form solution of the simplified optimization problem with box constraints can readily be derived [23]. This approach to calculate \( \alpha_{E,E'} \) is used in Zalesak's FCT limiter [4], which we employ in Algorithm 2 below. If \( u_{h}^\text{low} \) is a fairly good consistent approximation, one iteration is usually enough (which is reflected by our numerical results in Section 5.2.1). The use of \( u_{h}^\text{low} := u^{m-1}_h \) in Algorithm 1, Step 2 requires more iterations to avoid the local delay effect (cf. Figure 5) caused by redundant limiting in situations when positive and negative fluxes cancel out contrary to the above worst-case assumptions.

**Algorithm 2** (Zalesak’s FCT limiter with global bounds).

This algorithm seeks \( \alpha_{E,E'}^{(m)} \in [0, 1] \) for \( E, E' \in \mathcal{E}_h \) with \( \partial E \cap \partial E' \neq \emptyset \) such that
\[
\bar{u}^{(m)}_{h,E} = \bar{u}^{(m-1)}_{h,E} - \frac{\tau}{|E|} \sum_{E \cap E' \neq E} \alpha_{E,E'}^{(m)} \mathcal{H}^{(m-1)}_{E,E'}
\]
satisfies \( u_\ast \leq \bar{u}^{(m)}_{h,E} \leq u' \) for \( E \in \mathcal{E}_h \). For \( E \in \mathcal{E}_h \):

1. **Sum up inflow and outflow fluxes:**
\[
P^+_E := \tau \sum_{E' \cap E} \max \left\{ 0, -\mathcal{H}^{(m-1)}_{E,E'} \right\}, \quad P^-_E := \tau \sum_{E' \cap E} \min \left\{ 0, -\mathcal{H}^{(m-1)}_{E,E'} \right\}.
\]

Remark: We multiplied \( P^+_E \) and \( Q^+_E \) by \( \tau \) to provide a physical interpretation. Mass is flux times time—\( P^+_E \) is a measure for the mass that is to be created in \( E \) if no limiting is applied.

2. **Determine admissible upper and lower bounds:**
\[
Q^+_E := |E| \left( u' - \bar{u}^{(m-1)}_{h,E} \right), \quad Q^-_E := |E| \left( u_\ast - \bar{u}^{(m-1)}_{h,E} \right).
\]

Remark: Mass is concentration times volume—\( Q^+_E \) is a measure for the mass that can be stored in \( E \) without creating a mean-value overshoot.
3. **Compute element correction factors:**

\[ [0, 1] \ni \alpha_E^+ := \min \left\{ 1, \frac{Q_E^1}{P_E^1} \right\}. \]

Remark: The percentage of how much of mass \( P_E^1 \) is allowed.

4. **Compute flux-correction factors:** For \( E \ni E' \neq E \),

\[
\alpha_{E,E'}^{(m)} := \begin{cases} \min(\alpha_{E,E'}^-, \alpha_{E,E'}^+) & \text{if } \mathcal{H}_{E,E'}^{(m-1)} < 0 \quad \text{(positive flux from } E' \text{ to } E), \\ \min(\alpha_{E,E'}^-, \alpha_{E,E'}^-) & \text{if } \mathcal{H}_{E,E'}^{(m-1)} > 0 \quad \text{(positive flux from } E \text{ to } E'). \end{cases}
\]

Remark: Recall that \( \mathcal{H}_{E,E'}^{(m-1)} = -\mathcal{H}_{E,E'}^{(m-1)}. \)

**Remark 2 (Algorithm 2).**

Zalesak’s original limiter uses local bounds instead of the global bounds \( u_*, u' \). In the context of element averages, Step 2 reads in this case

\[
Q_E^x := |E| \left( u_E^x - \bar{u}_{h,E}^{(m-1)} \right), \quad Q_E^- := |E| \left( u_E^- - \bar{u}_{h,E}^{(m-1)} \right),
\]

where

\[
u_E^x := \min \left\{ \bar{u}_{h,E}^{(m-1)}, \min_{e \subseteq \Gamma_h} \bar{u}_{h,E'}^{(m-1)} \right\}, \quad u_E^- := \max \left\{ \bar{u}_{h,E}^{(m-1)}, \max_{e \subseteq \Gamma_h} \bar{u}_{h,E'}^{(m-1)} \right\}.
\]

\( \bar{u}_{h,E}^{(m)} \) then satisfy the local discrete MP

\[
u_{E}^x \leq \bar{u}_{h,E}^{(m)} \leq u_{E}^x
\]

for \( E \in E_h \). This algorithm was originally used as a one-step post-processing procedure. Its use in an iterative framework was proposed in [3, 39] to reduce the levels of numerical diffusion. If the predictor \( u_{h,E}^{\text{low}} \) is obtained with a consistent low-order method, improvements that can be achieved by performing more than one iteration or using less restrictive bounds are typically marginal. However, a non-iterative FCT scheme in which \( \bar{u}_{h,E} \) serves as the backup solution must use (nearly) optimal correction factors \( \alpha_{E,E'}^{\text{opt}} \), corresponding to the best bound-preserving flux approximation. Instead of solving the inequality-constrained optimization problem for \( \alpha_{E,E'}^{\text{opt}} \), we minimize the amount of flux limiting by using global bounds and performing as many iterations as it takes to recover a sufficiently accurate approximation to \( \alpha_{E,E'}^{\text{opt}} \mathcal{H}_{E,E'} \). If the target flux \( \mathcal{H}_{E,E'} \) is preconstrained by using slope-limited input states (see below) and does not create undershoots/overshoots, it will be recovered after sufficiently many iterations. We remark that the iterative FCT algorithm that we use in this paper differs from the one employed in [1]. It guarantees that each iterate is bound-preserving but does not generally produce \( \alpha_{E,E'}^{\text{opt}} = 1 \) and exit immediately if the initial guess is bound-preserving. The algorithm proposed in [1] does because negative fluxes are incorporated into the bounds for positive fluxes and vice versa. However, intermediate solutions are not provably bound-preserving in this version.

**4. Slope limiter — enforcing a global MP**

A slope limiter is a local post-processing filter that constrains linear and higher order parts of local solutions to satisfy a local MP pointwise. The local mean values are left unchanged by this procedure. Hence, the use of flux limiters may be required to guarantee the validity of the pointwise global MP (2). However, the numerical fluxes \( \mathcal{H}_{E,E'} \) are less likely to produce undershoots/overshoots if they are defined in terms of the well-behaved mean values or slope-limited DG approximations. In this work, we use slope limiters to enforce numerical admissibility conditions (local bounds) and flux limiters to enforce physical admissibility conditions (global bounds). As we show in the numerical study in Section 5.2.1, the deactivation of slope limiting imposes a high burden on the flux limiter, which may give rise to unacceptably large consistency errors in algorithms that produce \( \bar{u}_{h,E} \) in the case \( \alpha_{E,E'}^- = 0 \) or \( \alpha_{E,E'}^- = 0 \). Moreover, a numerical solution satisfying the global MP (2) may still exhibit spurious oscillations (within \( [u_*, u'] \)), whereas a flux limiter based on a local MP is likely to produce larger consistency errors due to more restrictive inequality constraints. The use of local bounds for slope limiting purposes is appropriate since it results in more reliable detection of nonphysical artifacts and does not produce additional consistency errors in the
DG approximation to the mean values. If the slope limiter works well, the flux limiter needs to be activated just in a small number of elements where nonphysical solution behavior is detected \textit{a posteriori}. Hence, the use of flux limiting may be restricted to these troubled elements, as in DG schemes based on the \textit{Multi-dimensional Optimal Order Detection} (MOOD) \cite{21, 22} methodology.

4.1. Linear vertex-based limiter

We illustrate the slope-limiting procedure using a linear slope limiter: Since \eqref{19} is applied after each time step, we suppress the time index \( n \) in \( u^*_E \) in this paragraph.

Assuming \( u_h \in \mathcal{P}_1(\mathcal{E}_h) \), there is the local representation

\[
  u_h|_E(x) = \overline{u}_{h,E} + \beta_E \nabla u_h(c_E) \cdot (x - c_E) \quad x \in E \in \mathcal{E}_h,
\]

with \( \beta_E = 1 \) where

\[
  c_E := \frac{1}{|E|} \int_E x \, dx
\]

is the centroid of \( E \). Clearly, \( \overline{u}_{h,E} = u_h(c_E) \). A slope limiter determines the maximum admissible slope for the linear reconstruction \eqref{19} by choosing admissible values \( \beta_E \in [0, 1] \) for \( E \in \mathcal{E}_h \) such that the MP \eqref{2} holds for all \( x \in E \) (recall that we assumed \eqref{1} to hold for element averages \( \overline{u}_{h,E} \)). The larger \( \beta_E \), the lesser the consistency error.

The above slope limiting methodology is called linear due to the representation \eqref{19}. If \( u_h \in \mathcal{P}_p(\mathcal{E}_h) \) for \( p \geq 2 \) is represented in a hierarchical basis, then \eqref{19} is obtained by setting any degrees of freedom associated with higher order degrees to zero and scaling the linear ones by \( \beta_E \). Hereby, \( \overline{u}_{h,E} \) remains unchanged, i.e., mass is conserved locally. Higher-order slope limiters are presented, e.g., in \cite{40}. Here, we present the linear vertex-based limiter of Kuzmin \cite{5}, which is an improved version of the Barth–Jespersen limiter \cite{41} by using a larger, vertex-oriented stencil (see \cite{18} for theoretical analysis that explains the poor performance of stencils based on edge/face neighborhoods). This limiter is independent of the element geometry and is applicable to our considered admissible grids, cf. Figure 1a. The linear vertex limiter is extendable to higher orders \cite{5, 42}.

For the vertex-based limiter, the correction factor \( \beta_E \) is chosen such that the above reconstruction is bounded in all vertices \( a_{E,i} \in E \) by the minimum and maximum integral means of elements that contain \( a_{E,i} \), i.e.,

\[
  \forall E \in \mathcal{E}_h, \quad u^E_{a,i} \leq u_h(a_{E,i}) \leq u^*_E,
\]

with

\[
  u^E_{a,i} := \min_{\{E \in \mathcal{E}_h | a_{E,i} \in E\}} \overline{u}_{h,E}, \quad u^*_E := \max_{\{E \in \mathcal{E}_h | a_{E,i} \in E\}} \overline{u}_{h,E}, \tag{21}
\]

cf. Figure 1b. Due to \( u_h|_E \in \mathcal{P}_1(E) \), the extrema of \( u_h|_E \) are attained at the vertices \( a_{E,i} \in E \). To enforce \eqref{20}, the correction factor \( \beta_E \) is defined as

\[
  \forall E \in \mathcal{E}_h, \quad \beta_E \defeq \min_i \begin{cases} u^E_{a,i} - \overline{u}_{h,E} & \text{if } u_{E,i} > u^*_E, \\ u^*_E - \overline{u}_{h,E} & \text{if } u^E_{a,i} \leq u_{E,i} \leq u^*_E, \\ u^E_{a,i} - \overline{u}_{h,E} & \text{if } u_{E,i} < u^*_E, \end{cases}
\]

where \( u_{E,i} \defeq u_h(a_{E,i}) \).

5. Application to the advection equation

5.1. Discretization notes

Consider the advection equation \eqref{1},\eqref{3} with possibly nonsolenoidal velocity field \( v \). Substituting discrete versions of \eqref{3a} and \eqref{8} into \eqref{7} and taking boundary elements into consideration, the DG formulation locally reads:
Find \( u_h(t) \in P_p(\mathcal{E}_h) \) such that for a.e. \( t \in J \),
\[
\forall \, w_h \in P_p(E), \quad \int_E w_h \partial_t u_h(t) - \int_E \nabla w_h \cdot \nu u_h(t) + \sum_{E \in \mathcal{E}} \alpha_{E,E'} \int_{e_{E,E'}} w_h_{|E} u_h^I(t) \nu \cdot n_E = 0 \tag{22a}
\]
with the upwind-sided trace
\[
u u_h^I_{|E,E'}(t,x) \coloneqq \begin{cases} u_h_{|E}(t,x) & \text{if } \nu(x) \cdot n_E \geq 0 \\ u_h_{|E'}(t,x) & \text{if } \nu(x) \cdot n_E < 0 \wedge x \notin \partial \Omega^m \\ u^m(\tau,x) & \text{if } x \in \partial \Omega^m \end{cases}
\]
where \( u_h(0) \) is obtained as the slope limited \( L^2 \) projection of \( u^0 \) into \( P_p(\mathcal{E}_h) \), i.e.,
\[
\forall \, w_h \in P_p(E), \quad \int_E w_h(0) - u^0 = 0. \tag{22b}
\]
For time discretization, we apply the time-implicit backward Euler method and the time-explicit forward Euler method. The backward Euler method yields a nonlinear problem due to the presence of \( \alpha_{E,E'} \) while for the forward Euler method a mass matrix needs to be inverted only.

### 5.2. Numerical results

All simulations in this paper are performed with the open source Matlab toolbox FESTUNG [43, 44].

#### 5.2.1. Solid body rotation scenario

![Initial data](image1.png) (a) initial data  ![Implicit, no limiter](image2.png) (b) implicit, no limiter  ![Implicit, FSL](image3.png) (c) implicit, FSL  ![Implicit, FSL and SL](image4.png) (d) implicit, FSL and SL

Figure 4: Projected and slope limited initial data \( u^0 \) of the solid body rotation scenario (a) using 14 006 elements. The color scale for \( u^0 \) in \([-1, 1]\) is blue to red, while values beyond are colored black. Solution at final time (after one rotation) using the backward Euler scheme and \( \tau = \frac{2\pi}{12 000} \) without limiting (b), with FSL (c), with FSL and SL (d).

As a first benchmark problem, we use the solid body rotation test proposed by LeVeque [45], which is often used to investigate slope limiter performance [42, 46]. For the discrete solution \( u^h \) of the time-discretized version of (22a) to satisfy a pointwise global MP, a slope limiter is sufficient. We use this scenario regardless to verify the MP for local means in the case a flux limiter (without slope limiter) is used.

At initial state, this scenario consists of a slotted cylinder, a sharp cone, and a smooth hump with centers \( c_1 \coloneqq (0.5, 0.75) \), \( c_2 \coloneqq (0.5, 0.25) \), and \( c_3 \coloneqq (0.25, 0.5) \), respectively, and radii \( r \coloneqq 0.15 \) on a square domain \( \Omega = (0, 1)^2 \) (cf. Figure 4). Let
\[
u^0(x) = \begin{cases} 1 & \text{if } |x - c_1| \leq r \wedge (x \leq 0.475 \vee x \geq 0.525 \vee y \geq 0.85) \\ 1 - \frac{1}{r}|x - c_2| & \text{if } |x - c_2| \leq r \\ \frac{1}{r(1 + \cos(\frac{\pi}{2}|x - c_3|))} & \text{if } |x - c_3| \leq r \\ 0 & \text{otherwise} \end{cases}
\]
with zero boundary data \( u^m = 0 \) and right-hand side \( f = 0 \). The three bodies are subjected to a time-independent, solenoidal velocity field \( \nu(x) \coloneqq [0.5 - y, x - 0.5]^T \) in a counterclockwise rotation over the time interval \( J \coloneqq (0, 2\pi] \).

Figure 4 shows the initial data \( u^0 \) and the solution after one rotation for explicit time stepping using \( \tau = \frac{2\pi}{12 000} \) without limiter, with flux, and with flux and SL active. For this time step size, usage of implicit time stepping, of
FSL(1) (non-iterative fractional step limiter), or of using a low-order scheme in the FSL make only insignificant difference. Hence, we use the larger step size of $\tau = \frac{2\pi}{5000}$ in Figure 5 for a comparison study: In all cases, the implicit time stepping yields more diffusive solutions. FSL(1) suppresses flux yielding a shrinking of the cylinder and the tip of the cone. Both locations are transported slower. The usage of the slope limiter yields a pointwise global MP solution, but introduces diffusion. If FSL(1) is used, the additional usage of a low order scheme (LOS) brings significant improvement. For FSL, using a LOS has no impact on the solution. The usage of ML without SL generates small limiting factors $\alpha_{E,E}$, which suppress the element-to-element transport. Since the inter-element transport is non-limited, this results in steep local gradients and eventually leads to divergence of the JFNK solver.
Figure 5: Stationary states of solid body rotation scenario and explicit Euler or implicit Euler time stepping with \( \tau = \frac{2\pi}{5000} \) (using \( \gamma_n = 1/(2\tau_n) \) for the ML on 14,006 elements). The color scale for \( \mathbf{u}_h \) in \([-1, 1]\) is blue to red, while values beyond are colored black. FSL(1) means that FSL only uses one single iteration and LOS means that FSL uses a low-order solution.

### 5.2.2. Implosion of a circle scenario

This test scenario, taken from [1], uses a non-solenoidal velocity field such that the continuous problem \{(1),(3)\} does not satisfy a MP. Even though being of academic type, this scenario covers situations that often occur in practice: When the velocity field is taken as discrete unknown from a flow simulation (e.g. Navier–Stokes equations), \( \nu_h \) might not be point-wise divergence free thus leading to unwanted compression effects accompanied by a violation of the discrete MP.
Let $\Omega = (0, 1)^2$ and let a stationary, non-solenoidal velocity field be given by

$$v(x) := \frac{c - x}{|c - x| + \delta} \quad \text{for} \quad x \in \overline{\Omega},$$

where $c := (0.5, 0.5)$ is the barycenter of $\Omega$, $| \cdot |$ the Euclidean vector norm, and $\delta := 10^{-12}$ a regularization parameter. The Dirichlet data $u^n$ is set to zero at $\partial \Omega^n$ and the initial data is

$$u^0(x) := \begin{cases} 0.5 & \text{if } |c - x| \leq 0.4, \\ 0 & \text{otherwise}. \end{cases}$$

The initial mass within the circle of radius 0.4 is transported toward the barycenter $c$ by $v$ and thus approaches the Dirac delta function. Constraining the solution by $u^* := 1$, the analytical stationary state is $u(x) = 1$ for $|c - x| \leq \sqrt{2}/5$ and $u(x) = 0$ elsewhere. The discretization parameters are $h = 1/128$ and $\tau_n = 10^{-3}$ and we set $u_* := 0$. As opposed to the solid body rotation scenario of Section 5.2.1, there is no low-order solution available here since the analytical solution does not satisfy the MP, i.e., the FSL uses $u^\text{low} = u^{n-1}$.

Snapshots of the simulation are shown in Figure 6. As expected, the unlimited solution $u^n$ tends toward the delta function with all mass concentrated at the center $c$ thus violating the DMP. Moreover, spurious oscillations occur near $c$ and at the boundary of the support of the unlimited solution leading to values outside of $[u^*, u^*]$. Application of ML or FSL forces the mean values $\overline{u}_{c,E}$ into the admissible bounds, while oscillations still prevent a pointwise MP from being attained. Similar to the solid body rotation scenario of Section 5.2.1 these oscillations are much stronger for the ML. When the SL is combined with the ML or the FSL, all oscillations are eliminated and a pointwise global MP is ensured. This demonstrates that in a scenario where a global MP does not hold for the analytical solution, both flux and slope limiting techniques need to be combined to ensure a pointwise global DMP.
Figure 6: Time-explicit simulation of the implosion scenario (using $\gamma_n = 1/(2\tau_n)$ for the ML on 14 006 elements). The color scale for $u^c_n$ in $[-1,1]$ is blue to red, while values beyond are colored black. Only when flux limiter are active, a MP for local means is met (b–e). A pointwise global MP is satisfied if in addition a SL is used (c,e). The ML is more oscillatory than the FSL.

6. Application to Cahn–Hilliard equation

6.1. Mathematical model

The phase segregation process modeled by the Cahn–Hilliard equation \{(1),(5)\} is driven by the dissipation of the energy functional

$$F := \int_\Omega \left( \frac{\kappa}{2} |\nabla u|^2 + \Psi(u) \right),$$

(23)

describing the Helmholtz free energy of the mixture [47, 48]. The parameter $0 < \kappa \ll 1$ in (23) is typically chosen as small as possible to generate a small interface width, but in numerical simulations $\kappa$ has to be bounded from below by the mesh size $h$ to guarantee a certain number of elements across the diffuse interface, e.g., $\kappa \geq h^2$ [49]. The order parameter $u$ can be chosen as difference of volume or mass fractions $u_A, u_B$ of either fluid, i.e., $u := u_A - u_B = 2u_A - 1$, where $u_A + u_B = 1$. The chemical potential $\phi$ as in (5b) is the variational derivative of $F$ with respect to $u$, i.e., $\phi = \delta_u F$. The boundary condition (5d) ensures that there is no flux across $\partial\Omega$ and (5c) enforces a 90 degree angle between the diffuse interface and $\partial\Omega$ in each time point. The flux $j$ in (5a) is scaled by the so-called mobility $M_k(u) := (\max(0, 1 - u^2))^k$. Typical choices are constant mobility ($k = 0$), degenerate mobility ($k = 1$), and biquadratic-degenerate mobility ($k = 2$).
The original expression of $\Psi$ proposed by Cahn and Hilliard [47] (often called logarithmic potential) has the shape of a double-well with two minima in $(-1, 1)$ if the (constant) system temperature is below the critical temperature of the mixture, which are parameters of the expression. In this case, a homogeneous mixture becomes energetically unfavorable and slightly miscible bulk compositions evolve. There are two frequently used expressions of $\Psi$ that have the local minima exactly at $u \in \{-1, 1\}$, cf. Figure 3: The polynomial potential

$$\Psi^{poly}(u) := \frac{1}{4}(u - 1)^2 (u + 1)^2 = \frac{1}{4}(1 + u^4) \cdot \frac{1}{2} u^2 =: \Psi^{poly}(u)$$

and the double obstacle potential

$$\Psi^{dob}(u) := \Psi^{dob}(u) := \begin{cases} \frac{1}{2} (1 - u^2) & \text{if } |u| \leq 1, \\ \infty & \text{otherwise}. \end{cases}$$

The latter, first proposed by Puri and Oono [50, 51], is the deep quench limit of the logarithmic potential.

Solutions $u$ of the Cahn–Hilliard equation $\{1, 5\}$ are physically meaningful in the interval $[-1, 1]$. For the degenerate mobility case $M_k$, $k \in \{1, 2\}$ or for usage of logarithmic or double obstacle potential, the MP (2) is fulfilled with $u_* = -1$, $u^* = 1$ [52], where the inequalities in (2) are strict for the logarithmic potential case. For constant mobility $M_0$ combined with the polynomial potential, exact solutions $u$ may violate the MP [8] and simulations suggest that solutions slightly violate this MP in bulk phases for space dimensions $d \in \{2, 3\}$ [49], cf. Figure 7 (a).

The standard DG discretization of $\{1, 5\}$ may fail to produce bound-preserving results due to the lack of control over numerical fluxes and slopes of numerical solutions. Violations of DMPs may also be caused by a poor choice of the time-stepping method or of the time step. In what follows, we apply the FSL of Section 3.3 to the DG discretization of $\{1, 5\}$ to enforce the DMP for constant mobility $M_0$ and the potentials $\Psi^{poly}$ and $\Psi^{dob}$. For $\Psi^{poly}$, consistency with the analytical solution (which does not satisfy the MP) is overridden by the requirement of physical consistency.

6.2. Discretization notes

Consider the Cahn–Hilliard equation $\{1, 5\}$ with constant mobility $M_0 = 1$. Substituting the discrete version of (10) into (11) and using $j_h = -\nabla \phi_h$ from (5d) yields the DG formulation:

Find $u_h(t) \in \mathbb{P}_p(\mathcal{E}_h)$ such that for a.e. $t \in J$,

$$\forall \ w_h \in \mathbb{P}_p(E), \quad \int_E w_h \cdot \partial_t u_h(t) + \int_E \nabla w_h \cdot \nabla \phi_h(t) + \sum_{E_h \cup E_h' = E} \alpha_{E,E'} \int_{E_h \cup E_h'} \left[ w_h |_{E} \cdot \nabla \phi_h(t) \cdot n_E + \frac{\varepsilon}{2} \nabla w_h |_{E} \cdot n_E \left[ \phi_h(t) \right] + \frac{\sigma}{h} w_h |_{E} \cdot n_E \left[ u_h(t) \right] \right] = 0, \tag{25a}$$

where $\phi_h(t) \in \mathbb{P}_p(\mathcal{E}_h)$ satisfies for a.e. $t \in J$,

$$\forall \ w_h \in \mathbb{P}_p(E), \quad \int_E w_h \cdot \nabla u_h(t) = \int_E \nabla w_h \cdot \nabla u_h(t) + \int_E w_h \cdot \Psi'(u_h(t)) + \kappa \sum_{E_h \cup E_h' = E} \int_{E_h \cup E_h'} \left[ - w_h |_{E} \cdot \nabla \phi_h(t) \cdot n_E + \frac{\varepsilon}{2} \nabla w_h |_{E} \cdot n_E \left[ u_h(t) \right] + \frac{\sigma}{h} w_h |_{E} \cdot n_E \left[ u_h(t) \right] \right] = 0, \tag{25b}$$

and where $u_h(0)$ is obtained by the $L^2$ projection of $u^0$ into $\mathbb{P}_p(\mathcal{E}_h)$, i.e.,

$$\forall \ w_h \in \mathbb{P}_p(E), \quad \int_E w_h \left( u_h(0) - u^0 \right) = 0. \tag{25c}$$

The (local) semi-discrete system ((25b), (25a)) holds in particular for boundary elements $E$, as $\nabla \phi_h = \nabla u_h = 0$ on $J \times \partial \Omega$ due to (5c) and (5d).
Remark 3 (Local mass conservation for Cahn–Hilliard with IPDG).
For the Cahn–Hilliard equation, substituting (5a) in (13) yields

$$\frac{d\bar{u}_E}{dt}(t) - \frac{1}{|E|} \int_{\partial E} \nabla \phi(t) \cdot n_E = 0$$

with \( \phi \) as defined in (5b). Choosing \( w_h = 1 \) and \( \alpha_{E,E'} = 1 \) in (25a) leads to

$$\frac{d\bar{u}_hE}{dt}(t) - \frac{1}{|E|} \sum_{E_0 \subsetneq E} \int_{E \setminus E_0} [\nabla \phi(t)] \cdot n_E + \frac{1}{|E|} \sum_{E_0 \subsetneq E} \int_{E \setminus E_0} \frac{\sigma}{h} \| \phi(t) \| \cdot n_E = 0$$

for a.e. \( t \in J \). Hence, the discrete problem mimics the continuous version up to penalty, cf. [12, Sec. 2.7.3]. Notice that the two formulations match when the jumps of \( u_h \) and of \( \phi_h \) vanish on element boundaries.

For time-discretization, we use the well-established convex–concave splitting method [53]: Let \( \Psi = \Psi_+ + \Psi_- \) be a decomposition of \( \Psi \) with a convex part \( \Psi_+ \) and a concave part \( \Psi_- \), which (non-uniquely) exists for \( \Psi \in C^2([-1, 1]) \). The decomposition of \( \Psi^{\text{poly}} \) and \( \Psi^{\text{dol}} \) is indicated in (24). System (25) is then discretized with the backward Euler method except of the term \( \Psi_+'(u_0(t)) \), which is treated explicit in time. This IMEX scheme is known to yield uniquely solvable and unconditionally energy stable time-discretizations of the (space-continuous) Cahn–Hilliard problem, i.e., \( \forall n \in \{1, \ldots, N_d\}, \quad F(u^n) \leq F(u^{n-1}) \) [52]. For \( \Psi = \Psi^{\text{poly}} \) as in (24a) and the SIP-DG case (\( \epsilon = -1 \)), unconditionally dissipation of discrete energy \( F_h \) was shown in [54]. Clearly, this IMEX scheme yields a nonlinear system in each time step if \( \Psi_- \) is nonlinear, which is the case for \( \Psi^{\text{poly}} \).

The global formulation of the fully discrete problem and the equivalent nonlinear matrix system without flux-correction factors \( \alpha_{E,E'} \) is stated in [55]. Due to the strong nonlinearity of \( \alpha_{E,E'} \) in the ML case, we use a Jacobian-free Newton–Krylov method [37, 38].

6.3. Numerical results

6.3.1. Spinodal decomposition scenario

Spinodal decomposition is the process of phase transformation from an initially homogeneous mixture into separate zones rich in either of the components and separated by a diffuse interface. In physical systems, it is initiated, for instance, by a spontaneous pressure or temperature drop. The spinodal decomposition scenario is a well-known numerical benchmark for Cahn–Hilliard discretizations, cf. e.g. [8, 56, 57, 58, 59]. Analytical solutions of the Cahn–Hilliard equation dissipate the total Helmholtz free energy \( F \) (cf. (23)) in the sense that

$$\frac{d}{dt} F(u(t)) \leq 0$$

for all time points \( t \in J \) under the constraint of mass conservation. It is desirable that numerical methods mimic these properties in a discrete way. Instead of a homogeneous mixture, the numerical setting required a “seed” and so \( u^0 \) is usually taken as perturbation around a value. As initial data, we define \( u^0 \) as slope-limited \( L^2 \) projection of a function whose values are quadrature point-wise randomly chosen from the set \([-0.99, 0, 0.99] \). The discretization parameters are

\[
\tau = 10^{-4}, \quad h = 2^{-6}, \quad \epsilon = 1 \quad \text{(NIPG)} \quad \text{and} \quad \kappa \text{ as indicated in the figures.}
\]

Figure 7 shows the non-limited DG solution for the polynomial potential case \( \Psi = \Psi^{\text{poly}} \), in which the analytical solution does not satisfy a MP. As expected, \( u^0_h \) exceeds the interval \( [u_-, u^+] \) on either side in some locations of the bulk phases. Application of the FSL forces the mean values \( \bar{u}^0_{h,E} \) into the admissible bounds. Notice that the elementwise linear solution still violates the pointwise DMP. Eventually, using both FSL and SL gives a solution that satisfies a global DMP pointwise. The FSL has the property to limit as less flux as possible, and indeed, the limiting procedure does not change the physics of the simulation.

Analytical solutions satisfy a global MP if \( \Psi = \Psi^{\text{dol}} \) is set. The IMEX scheme described in Section 6.2 applied to the semi-discrete DG problem yields a sequence of linear problems, as \( \Psi^{\text{dol}} \) is concave and thus is treated explicit in time. Application of the non-limited DG scheme produces discrete solutions that locally violate the DMP after some time steps—even if FSL is used (14 time steps satisfy the DMP in Figure 8 (a), for instance). As in that case, \( \Psi = \infty \) at some quadrature points, \( \Psi' \) is not defined and hence the simulation terminates with failure. Using however both FSL and SL as post-processor after each time step, the simulation succeeds, cf. Figure 8. As an aside, the initial data \( u^0_h \) is the same as in Figure 7. The different choices of potentials have a high impact on the physics.
7. Conclusion

In this paper, we propose two flux-limiting schemes applicable to a wide family of DG schemes for scalar conservation laws: a monolithic flux limiter and a fractional step limiter. Both limiters enforce a discrete maximum principle for element mean values with global upper and lower bounds. While the first limiter is a promising tool for numerical analysis, where $L^\infty$ bounds are required, the latter is a cost-effective local post-processing filter for
practical simulations. Either of the flux limiters act locally, i.e., they leave discrete solutions unchanged in non-
troubled regions. Pointwise maximum principles are attained if a subsequent slope-limiter is applied. The imposition
of global bounds on element averages does not degrade the rates of convergence to analytical solutions that lie within
these bounds.

Due to its simplicity, general applicability, and local post-processing nature, we hope that the fractional step lim-
iter will be used for a broad range of hyperbolic and parabolic conservation laws in the DG community, in particular,
in large-scale computing.

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( | \cdot | )</td>
<td>Average on a face ( \Gamma_h ) ( \equiv e = \partial E \cap \partial E' ), ( | w | \equiv (w_E + w_{E'})/2 ).</td>
</tr>
<tr>
<td>( | \cdot | )</td>
<td>Jump on a face ( \Gamma_h ) ( \equiv e = \partial E \cap \partial E' ), ( | w | \equiv w_E n_E + w_{E'} n_{E'} ).</td>
</tr>
<tr>
<td>( a_{E,i} )</td>
<td>Vertex ( i ) of element ( E ).</td>
</tr>
<tr>
<td>( c_E )</td>
<td>Centroid of element ( E ), ( c_E \equiv \frac{1}{</td>
</tr>
<tr>
<td>( e_{E,E'} )</td>
<td>Symmetrization parameter, ( \epsilon = -1 ) (SIP), ( \epsilon = 0 ) (IIP), ( \epsilon = 1 ) (NIP).</td>
</tr>
<tr>
<td>( E )</td>
<td>Convex cell/element ( E \in \mathcal{E}_h ) with Lipschitz boundary ( \partial E ).</td>
</tr>
<tr>
<td>( \mathcal{E}_h )</td>
<td>Set of elements, ( \mathcal{E}_h \equiv { E }, \mathbb{R}^d \supset \overline{\Omega} \equiv \bigcup E ).</td>
</tr>
<tr>
<td>( F )</td>
<td>Helmholtz free energy as a functional of ( u ), cf. (23).</td>
</tr>
<tr>
<td>( \Gamma_h )</td>
<td>Union of interior faces, ( \Gamma_h \equiv \bigcup_{E \in \mathcal{E}_h} \partial E ).</td>
</tr>
<tr>
<td>( \mathcal{H}_{E,E'} )</td>
<td>Net mass flux from ( E ) into ( E' ) across ( \partial E \cap \partial E' ), cf. (12).</td>
</tr>
<tr>
<td>( j )</td>
<td>Flux of conservation law (1).</td>
</tr>
<tr>
<td>( J )</td>
<td>Time interval, ( J := (0, T] ).</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>Constant interface parameter.</td>
</tr>
<tr>
<td>( M )</td>
<td>Constant or degenerate mobility.</td>
</tr>
<tr>
<td>( n_E )</td>
<td>Unit normal on ( \partial E ) outward of ( E \in \mathcal{E}_h ).</td>
</tr>
<tr>
<td>( N_d )</td>
<td>Number of time steps until ( T ) is reached.</td>
</tr>
<tr>
<td>( \Omega )</td>
<td>Spatial domain, ( \Omega \subset \mathbb{R}^d ), ( d \in {2, 3} ) (open, connected, bounded, Lipschitz boundary ( \partial \Omega )).</td>
</tr>
<tr>
<td>( P_p(E) )</td>
<td>Space of polynomials of degree at most ( p ) on element ( E ), ( p \in \mathbb{N}_0 ).</td>
</tr>
<tr>
<td>( P_p(\mathcal{E}_h) )</td>
<td>( \bigcap_{E \in \mathcal{E}_h} P_p(E), p \in \mathbb{N}_0 ).</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>Jump-penalty parameter, ( \sigma &gt; 0 ), depends on choice of ( \epsilon ).</td>
</tr>
<tr>
<td>( u )</td>
<td>Exact solution of the classical or the weak problem (1).</td>
</tr>
<tr>
<td>( u_h )</td>
<td>Time-dependent, space-discrete solution.</td>
</tr>
<tr>
<td>( u^p_h )</td>
<td>Fully discrete solution at time level ( n ).</td>
</tr>
<tr>
<td>( \overline{u}_E )</td>
<td>Integral mean of ( u ) on ( E ), ( \overline{u}_E \equiv \frac{1}{</td>
</tr>
<tr>
<td>( \text{DG} )</td>
<td>Discontinuous Galerkin.</td>
</tr>
<tr>
<td>( \text{DM} )</td>
<td>Discrete maximum principle.</td>
</tr>
<tr>
<td>( \text{FCT} )</td>
<td>Flux corrected transport.</td>
</tr>
<tr>
<td>( \text{SL} )</td>
<td>Slope limiter.</td>
</tr>
<tr>
<td>( \text{FSL} )</td>
<td>Iterative fractional-step flux limiter.</td>
</tr>
<tr>
<td>( \text{LOS} )</td>
<td>Low-order scheme.</td>
</tr>
<tr>
<td>( \text{ML} )</td>
<td>Monolithic limiter.</td>
</tr>
<tr>
<td>( \text{MP} )</td>
<td>Maximum principle.</td>
</tr>
</tbody>
</table>

References


382 M. Herz, P. Knabner: Modeling and simulation of coagulation according to DVLO-theory in a continuum model for electrolyte solutions  (16.05.2014)
383 M. Herz, P. Knabner: A thermodynamically consistent model for multicomponent electrolyte solutions  (02.06.2014)
384 M. Gahn, P. Knabner, M. Neuss-Radu: Homogenization of reaction-diffusion processes in a two-component porous medium with a nonlinear flux condition at the interface, and application to metabolic processes in cells  (09.07.2014)
385 J. Jahn: Directional derivatives in set optimization with the set less order relation  (08.08.2014)
386 N. Ray, P. Knabner: Upscaling flow and transport in an evolving porous medium with general interaction potentials  (06.10.2014)
387 J. Greifenstein, M. Stingl: Simultaneous material and topology optimization  (21.10.2014)
388 R. Schulz, N. Ray, F. Frank, H. Mahato, P. Knabner: Strong solvability up to clogging of an effective diffusion-precipitation model in an evolving porous medium  (06.02.2015)
389 E. Bänsch, A. Brenner: A-posteriori estimates for the rotational pressure-correction projection method  (29.05.2015)
390 M. Bischoff, J. Jahn: Economic objectives, uncertainties and decision making in the energy sector  (07.10.2015)
391 R. Schulz, P. Knabner: Derivation and analysis of an effective model for biofilm growth in evolving porous media  (08.03.2016)
392 M. Gahn, M. Neuss-Radu: A characterization of relatively compact sets in $L^p(\Omega, B)$  (29.04.2016)
394 J. Hoffmann, S. Kräutle, P. Knabner: Existence and uniqueness of a global solution for reactive transport with mineral precipitation-dissolution and aquatic reactions in porous media  (15.11.2016)
396 M. Gahn, M. Neuss-Radu, P. Knabner: Derivation of effective transmission conditions for domains separated by a membrane for different scaling of membrane diffusivity  (22.03.2017)
397 M. Gahn, M. Neuss-Radu, P. Knabner: Effective transmission conditions for processes through thin heterogeneous layers with nonlinear transmission at the bulk-layer interface  (24.03.2017)
398 N. Ray, R. Schulz: Derivation of an effective dispersion model for electroosmotic flow involving free boundaries in a thin strip  (31.05.2017)
399 A. Rupp, K.U. Totsche, A. Prechtel, N. Ray: Discrete-continuum multiphase model for structure formation in soils including electrostatic effects  (28.05.2018)