Global existence of weak solutions of a model for electrolyte solutions – Part 2: Multicomponent case

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Abstract

This paper analytically investigates the Darcy–Poisson–Nernst–Planck system. This system is a mathematical model for electrolyte solutions. In this paper, we consider electrolyte solutions, which consist of a neutral fluid and multiple suspended charged chemical species with arbitrary valencies. We prove global existence and uniqueness of weak solutions in two space dimensions.

Keywords: Global existence, electrolyte solution, electrohydrodynamics, Moser iteration, generalized Schauder fixed point theorem, Darcy–Poisson–Nernst–Planck system.

1 Introduction

From the introduction of Part 1 of this work, we know that many complicated phenomena in hydrodynamics and biology can be modeled as electrolyte solutions, as these models simultaneously capture the following three ubiquitous processes: (i) The transport of the charged particles, (ii) the hydrodynamic fluid flow, (iii) the electrostatics.

Darcy–Poisson–Nernst–Planck systems are classical models for electrolyte solutions in case of laminar flow in porous media. A detailed derivation of these systems are given in [3, 12, 18, 22–25, 29, 30, 32]. In Part 1 of this work, we mentioned, that among many others, mathematical models for electrolyte solutions have been investigated analytically in [6, 9, 11, 13–17, 21, 23, 26, 31, 32, 34].

In particular, existence of solutions for electrolyte solutions, which contain multiple charged solutes, were proven amongst others in [6, 9, 15, 26, 27]. More precisely, the authors of [6] proved local in time existence for a fluid-particle system. The existence results of [26, 27] were proven under the additional assumption of a volume-additivity constraint and by including an additional reaction force term in the transport equations. These additional assumptions allow to bypass in [26, 27] the below mentioned serious difficulties. [9] dealt with a stationary system and in [15], existence in two dimensions was established for electrolyte solution at rest, which may involve nonlocal constraints in the equation for the electrostatic potential.

In case of multiple charged chemical species with arbitrary valencies, the main difficulty is to prove a priori estimates for the chemical species, which are valid on arbitrary large time intervals and which are independently of the electric field. We briefly sketch this difficulty by firstly considering the case of two oppositely charged species $c_1$ (positively charged) and $c_2$ (negatively charged) with valencies $z_1 > 0 > z_2$. In Part 1 of this work, we proposed to use in the proof of a priori estimates for $c_1$ the weighted test functions $\varphi = |z| \phi$. Together with Gauss’s law, i.e. $\nabla \cdot E = z_1 c_1 - |z_2| c_2$, we thereby obtained the following pointwise sign condition for the sum of the “electric drift integrals”, which describe the
electrophoretic motion of the $c_i$

$$-2 \sum_l \langle z_l c_l \nabla (|z_l| c_l) \rangle_{L^2(\Omega)} = \left( \nabla \cdot E, (z_l c_l)^2 - (|z_l| c_l)^2 \right)_{L^2(\Omega)}$$

$$= \left( z_l c_l - |z_l| c_l, (z_l c_l)^2 - (|z_l| c_l)^2 \right)_{L^2(\Omega)} \geq 0, \quad \text{as } [a-b] [a^2-b^2] \geq 0 \text{ for } a, b \geq 0.$$  

Due to this pointwise sign condition, we omitted the sum of the “electric drift integrals” in the proof of a priori estimates for the chemical species $c_i$. However, in the multicomponent case, i.e., in case of of $L \in \mathbb{N}$ charged solutes $c_i$ with arbitrary valencies $z_i$, the corresponding sum of these integrals reads with Gauss’s law $\nabla \cdot E = \sum_l z_l c_l$ as

$$-2 \sum_l \langle \text{sign}(z_l) |z_l| c_l E, \nabla (|z_l| c_l) \rangle_{L^2(\Omega)} = \left( \nabla \cdot E, \sum_l \text{sign}(z_l) (|z_l| c_l)^2 \right)_{L^2(\Omega)}$$

$$= \left( \sum_l \text{sign}(z_l) |z_l| c_l, \sum_l \text{sign}(z_l) (|z_l| c_l)^2 \right)_{L^2(\Omega)} \iff I.1 = I.2 = I.3.$$  

It is easy to show that $I.3$ does not satisfy a pointwise sign condition. Thus, we now have to estimate one of the integrals $I.1$–$I.3$ by suitable norms of the integrands. This can easily be done with standard arguments. More precisely, we obtain with Hölder’s inequality, Gagliardo-Nirenberg’s inequality, and Young’s inequality the estimates (space dimension $n = 2, 3$)

(a) $I.3 \leq C \sum_l \|c_l\|^2_{L^2(\Omega)} \leq \delta \sum_l \|\nabla c_l\|^2_{L^2(\Omega)} + \delta^{-1} C \sum_l \|c_l\|^6_{L^2(\Omega)},$

(b) $I.2 \leq \delta \sum_l \|\nabla c_l\|^2_{L^2(\Omega)} + \delta^{-1} C \|\nabla \cdot E\|^4/(4-n)_{L^2(\Omega)} \sum_l \|c_l\|^2_{L^2(\Omega)},$

(c) $I.1 \leq \delta \sum_l \|\nabla c_l\|^2_{L^2(\Omega)} + \delta^{-1} C \|E\|^4_{L^2(\Omega)} \sum_l \|c_l\|^2_{L^2(\Omega)}.$

Provided we have an a priori bound for the electric field or its divergence, which is independent of $c_l$, we are done by choosing one of the estimates (b) or (c). However, in Poisson–Nernst–Planck systems the equations for the electric field and its divergence depend on $c_l$, which leads to the a priori bound $\|E\|_{L^2(\Omega)} + \|\nabla \cdot E\|_{L^2(\Omega)} \leq C \sum_l \|c_l\|_{L^2(\Omega)}$. Substituting this bound into (b) or (c) shows that these estimates lead to (a). Finally, the proof of a priori estimates for $c_l$ in combination with (a) results in

$$\frac{d}{dt} \sum_l \|c_l\|^2_{H^1(\Omega)} + \sum_l \|\nabla c_l\|^2_{L^2(\Omega)} \leq C \sum_l \|c_l\|^2_{L^2(\Omega)} + C \sum_l \|c_l\|^6_{L^2(\Omega)}.$$

Hence, we can apply the standard version of Gronwall’s inequality, which we need to obtain a uniform a priori bound from the preceding equation. In fact, we just have a nonlinear version due to Willett-Wong, cf. [33], which is in our case applicable just on a very small time interval, cf. [5, Theorem 4.9]. An iterative application of this nonlinear Gronwall’s inequality does not lead to an extension to arbitrary large time intervals $[0, T_0]$, as the constants may blow up in finite time.

In this paper, we propose to substitute Gauss’s law $\nabla \cdot E = \sum_l z_l c_l =: \rho_f$ into the above estimate (b). Thereby, estimate (b) reads as

$I.2 \leq \delta \sum_l \|c_l\|^2_{H^1(\Omega)} + \delta^{-1} C \|\rho_f\|^4/(4-n)_{L^2(\Omega)} \sum_l \|c_l\|^2_{L^2(\Omega)}.$

It now remains to show a uniform bound for $\rho_f$ in $L^{4/(4-n)}(I: L^2(\Omega))$, which is the crucial task in the current chapter. We successfully establish such a uniform bound by means of entropy estimates.

---

1 Consider, e.g., three chemical species with valencies $z_1 = z_2 = 1$, $z_3 = -1$. Assume that the values at a given point in space and time are $c_1 = c_2 = 1, c_3 = \sqrt{3}$. This gives $(1 + 1 - \sqrt{3})(1 + 1 - 3) = -(2 - \sqrt{3}) < 0.
which lead to bounds for certain Lyapunov functions and which have been previously used in a different context, e.g., in [19]. However, these techniques lead to a uniform bound for $\rho_f$ in $L^2(I; L^2(\Omega))$. This is the reason why the presented existence result is restricted to two space dimensions.

The contribution of this paper is to show in two space dimensions the global existence, uniqueness, and boundedness of weak solutions for electrolyte solutions that consist of a neutral fluid and multiple charged solutes with arbitrary valencies. Due to the entropy estimates we can avoid further restrictions, such as the electroneutrality constraint, cf. [3], or the volume-additivity constraint, cf. [26]. Thus, the presented results apply to two dimensional models of general electrolyte solutions, which are captured by the Darcy–Poisson–Nernst–Planck system.

The rest of this paper is organized as follows: In Section 2, we present the model equations and we prove that solutions are unique. In Section 3, we introduce the fixed point method, and in Section 4.1, we prove the crucial a priori estimates. Finally, in Section 4.2, we show the global existence. Note, that we omit several proofs in Section 4.1 and Section 4.2, as we wrote Part 1 of this work such that it allows us to copy the coinciding parts of the proof.

## 2 Model Equations

Henceforth, we use the notation (N1)–(N4), which we already introduced in Part 1 of this work. Furthermore, in Part 1 of this work, we pointed out that Darcy–Poisson–Nernst–Planck systems consist of the following three coupled conservation laws:

**Law 1 – Gauss’s law**

\[
\mathcal{E}^{-1}E = -\nabla \Phi \quad \text{in } \Omega_T, \tag{2.1a}
\]

\[
\nabla \cdot E = \rho_f + \rho_b \quad \text{in } \Omega_T, \tag{2.1b}
\]

\[
\rho_f = \theta \sum_l z_lc_l \quad \text{in } \Omega_T, \tag{2.1c}
\]

\[
E \cdot \nu = \sigma^{-1} \quad \text{on } \Gamma_T. \tag{2.1d}
\]

**Law 2 – Darcy’s law**

\[
K^{-1}u = \mu^{-1} (-\nabla p + \mathcal{E}^{-1} \rho_f E) \quad \text{in } \Omega_T, \tag{2.1e}
\]

\[
\nabla \cdot u = 0 \quad \text{in } \Omega_T, \tag{2.1f}
\]

\[
u \cdot u = f \quad \text{on } \Gamma_T. \tag{2.1g}
\]

**Law 3 – Nernst-Planck equations**

\[
\theta \partial_t c_l + \nabla \cdot \left( D \nabla c_l + c_l [u + ez_l (e_k T)^{-1} E] \right) = \theta R_l(c) \quad \text{in } \Omega_T, \tag{2.1h}
\]

\[
(D \nabla c_l + c_l [u + ez_l (e_k T)^{-1} E]) \cdot \nu = 0 \quad \text{on } \Gamma_T, \tag{2.1i}
\]

\[
c_l(0) = c_{0,l} \quad \text{on } \Omega. \tag{2.1j}
\]

**Remark 2.1.** The homogeneous boundary flux in equation (2.1i), can be equivalently expressed with equations (2.1d) and (2.1g) by

\[
(D \nabla c_l) \cdot \nu = -c lf - (e_k T)^{-1} ez_l c_\sigma \quad \text{on } \Gamma_T.
\]

Thus, the homogeneous boundary flux condition is equivalent to a homogeneous Robin boundary condition for the diffusion part. □

### 2.1 Weak formulation of the model

Firstly, we introduce the required notation for the analytical investigations.
(N5) **Spaces:** For $k > 0, p \in [1, \infty]$, we denote the Lebesgue spaces for scalar-valued and vector-valued functions by $L^p(\Omega)$ and the respective Sobolev spaces by $W^{k,p}(\Omega)$, cf. [2]. Furthermore, we set $H^k(\Omega) := W^{k,2}(\Omega)$ and we refer for the definition of the Bochner spaces $L^p(I; V)$, $H^p(I; V)$ over a Banach space $V$ to [28]. The $H^1_0(\text{div}; \Omega)$-spaces are defined, e.g., in [8] by $H^1_0(\text{div}; \Omega) := \{ v \in H^1(\Omega) : \nabla \cdot v \in H^1(\Omega), \nabla \cdot v = f \text{ on } \Omega \}$.

(N6) **Products:** We denote by $(\cdot, \cdot)_H$ the inner product on a Hilbert space $H$ and by $(\cdot, \cdot)_{V \times V}$ the dual pairing between a Banach space $V$ and its dual space $V^*$. On $\mathbb{R}^n$, we just write $v \cdot u := (v, u)_{\mathbb{R}^n}$ and on $L^2(\Omega; \mathbb{R}^d)$, we just denote $(\cdot, \cdot)_\Omega := (\cdot, \cdot)_{L^2(\Omega; \mathbb{R}^d)}$. In particular the dual pairing between $H^1(\Omega)$ and its dual $H^{-1}(\Omega)^*$, we abbreviate by $(\cdot, \cdot)_{H^1(\Omega)^* \times H^{-1}(\Omega)}$.

Secondly, we henceforth impose the following assumptions:

(A1) **Geometry:** Let $n = 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, i.e. $\Gamma \in C^{0,1}$.

(A2) **Initial data:** The initial data $c_{0,I}$ are non-negative and bounded, i.e.,

$$0 \leq c_{0,I}(x) \leq M_0 \text{ for a.e. } x \in \Omega \text{ for some } M_0 \in \mathbb{R}_+.$$

(A3) **Ellipticity:** The diffusivity tensor $D$ and the permeability tensor $K$ satisfy

$$\nabla \cdot \xi > \alpha_D |\xi|^2 \text{ and } K^{-1} \cdot \xi > \alpha_K |\xi|^2 \text{ for all } \xi \in \mathbb{R}^n,$$

$$\nabla \cdot \eta \leq C_D |\eta| \text{ and } K^{-1} \cdot \eta \leq C_K |\eta| \text{ for all } \eta, \xi \in \mathbb{R}^n.$$

(A4) **Coefficients:** The porosity $\theta$, the dynamic viscosity $\mu$, and the electric permittivity $\epsilon$ are positive constants.

(A5) **Reaction rates:** The reaction rate functions $R_i : \mathbb{R} \to \mathbb{R}$ are global Lipschitz continuous functions, i.e., $R_i \in C^{0,1}(\mathbb{R})$ with Lipschitz constant $C_{R_i}$. Furthermore, we assume $R_i(0) = 0$ and $R_i(v_1, \ldots, v_L) \geq 0$ for all $v \in \mathbb{R}^L$ with $v_i \leq 0$. This means, in case a chemical species vanishes, it can only be produced.

(A6) **Boundary data:** We assume $\sigma \in L^\infty(\Gamma_T)$ and $f \in L^\infty(\Gamma_T)$. Furthermore, we suppose that functions $f, \sigma \in L^\infty(I; W^{1,\infty}(\Omega))$ with $\sigma \cdot c = c$ and $f \cdot c = f$ exist.

(A7) **Background charge density:** We assume $\rho_0 \in L^\infty(\Gamma_T)$.

Thirdly, we define the weak formulation of the Darcy–Poisson–Nernst–Planck system.

**Definition 2.2** (Weak solution). The vector $(E, \Phi, u, p, c) \in \mathbb{R}^{2+2n+L}$ is a weak solution of the Darcy–Poisson–Nernst–Planck system (2.1a)–(2.1j), if and only if

(i) $(E, \Phi) \in L^\infty(I; H^1_0(\text{div}; \Omega)) \times L^\infty(I; L^2(\Omega))$ solve for all $(v, \varphi) \in H^1_0(\text{div}; \Omega) \times L^2(\Omega)$

$$\begin{align*}
\left(\varepsilon^{-1} E, v\right)_\Omega &= \left(\Phi, \nabla \cdot v\right)_\Omega, \\
\left(\nabla \cdot E, \varphi\right)_\Omega &= \left(\rho_0 + \rho_f, \varphi\right)_\Omega.
\end{align*}
$$

(ii) $(u, p) \in L^\infty(I; H^1_0(\text{div}; \Omega)) \times L^\infty(I; L^2(\Omega))$ solve for all $(v, \varphi) \in H^1_0(\text{div}; \Omega) \times L^2(\Omega)$

$$\begin{align*}
\left(K^{-1} u, v\right)_\Omega &= \left(\mu^{-1} p, \nabla \cdot v\right)_\Omega + \left(\mu^{-1} \varepsilon^{-1} \rho_f, v\right)_\Omega, \\
\left(\nabla \cdot u, \varphi\right)_\Omega &= 0.
\end{align*}
$$

(iii) $c_l \in L^\infty(I; L^2(\Omega)) \cap L^2(I; H^1(\Omega)) \cap H^1(I; H^1(\Omega)^*) \cap L^\infty(\Gamma_T)$ solves for all $\varphi \in H^1(\Omega)$ and for $l = 1, \ldots, L$

$$\langle \partial_t c_l, \varphi \rangle_{1,\Omega} + \langle D \nabla c_l, \nabla \varphi \rangle_{\Omega} - \left( c_l u + e z_l (\epsilon k_B T)^{-1} E \right) \cdot \nabla \varphi_{\Omega} = \langle \theta R_l(c), \varphi \rangle_{\Omega},$$

and $c_l$ take its initial values in the sense that

$$\lim_{t \to 0} (c_l(t) - c_{0,l}, \varphi)_{\Omega} = 0 \quad \text{for all } \varphi \in L^2(\Omega).$$

**Remark 2.3.** As already showed in Part 1 of this work, equations (2.2e) are not well-defined without having $c_l \in L^\infty(\Gamma_T)$. Thus $c_l \in L^\infty(\Gamma_T)$ is mandatory for a well-defined weak formulation.
2.2 Uniqueness

We note that we can skip some of the following proofs, as we wrote Part 1 of this work such that we can directly copy some proofs.

**Theorem 2.4** (Uniqueness). Let (A1)–(A7) be valid and let \((E, \Phi, u, p, c) \in \mathbb{R}^{2+2n+L}\) be a weak solution of (2.1a)–(2.1) according to Definition 2.2. Then, \((E, \Phi, u, p, c)\) is unique.

*Proof.* The proof is identical to the corresponding proof of Part 1 of this work. Furthermore, the proof holds even in three space dimensions. \(\square\)

3 Fixed Point Operator

We apply the same fixed point approach as already used in Part 1 of this work. For more details concerning this approach, we refer to Part 1 of this work.

**Definition 3.1** (Fixed point operator). Let \(K \subset X\) be a subset of the Banach space \(X\), which is given by \(X := [L^\infty(I; L^2(\Omega))] \cap L^2(I; H^1(\Omega)) \cap H^1(I; H^1(\Omega)^*) \cap L^\infty(\Omega^2)]^L\). We introduce the fixed point operator \(F\) by

\[
F := F_3 \circ F_2 \circ F_1 : K \subset X \rightarrow X.
\]

Herein, the suboperator \(F_1\) is defined by

\[
F_1 : \left\{ \begin{array}{l}
K \rightarrow X \times L^\infty(I; H_0(div; \Omega)) \times L^\infty(I; L^2(\Omega)/\mathbb{R}) =: Y \\
(c, c, E, \Phi) \mapsto (c, E, \Phi), \text{ with } (E, \Phi) \text{ solving for all } (v, \varphi) \in H_0(div; \Omega) \times L^2(\Omega)
\end{array} \right.
\]

\[
(\mathcal{E}^{-1}E, v)_\Omega = (\Phi, \nabla \cdot v)_\Omega,
\]

\[
(\nabla \cdot E, \varphi)_\Omega = (\rho_l + \rho_f, \varphi)_\Omega,
\]

with \(\rho_f\) defined ind (2.1c).

Furthermore, the suboperator \(F_2\) is defined by

\[
F_2 : \left\{ \begin{array}{l}
Y \rightarrow Y \times L^\infty(I; H_f(div; \Omega)) \times L^\infty(I; L^2(\Omega)/\mathbb{R}) =: Z \\
(c, E, \Phi, u, p) \mapsto (c, E, \Phi, u, p), \text{ with } (u, p) \text{ solving for all } (v, \varphi) \in H_0(div; \Omega) \times L^2(\Omega)
\end{array} \right.
\]

\[
(K^{-1}u, v)_\Omega = (\mu^{-1}p, \nabla \cdot v)_\Omega + (\mu^{-1}\mathcal{E}^{-1}\rho_f E, v)_\Omega,
\]

\[
(\nabla \cdot u, \varphi)_\Omega = 0.
\]

Finally, the suboperator \(F_3\) is defined by

\[
F_3 : \left\{ \begin{array}{l}
Z \rightarrow X \\
(c, E, \Phi, u, p) \mapsto c = (c_1, c_2), \text{ with } c_1 \text{ solving for all } \varphi \in H^1(\Omega) \text{ and } l = 1, \ldots, L
\end{array} \right.
\]

\[
(\theta \partial_t c_l, \varphi)_\Omega + (D \nabla c_l, \nabla \varphi)_\Omega - (c_l[u + c_l(e_k T)^{-1}E], \nabla \varphi)_\Omega
\]

\[
= (\theta R_l(e), \varphi)_\Omega,
\]

and \(c_l\) take its initial values in the sense that

\[
\lim_{t \searrow 0} (c_l(t) - c_{e_0,l}, \varphi)_\Omega = 0 \quad \text{for all } \varphi \in L^2(\Omega)\).
\]

\(\square\)

**Lemma 3.2** (well-definedness). Let (A1)–(A7) be valid. Then, the operator \(F : K \subset X \rightarrow X\) defined in Definition 3.1, is well-defined.

*Proof.* The proof is identical to the corresponding proof of Part 1 of this work. Furthermore, the proof holds even in three space dimensions. \(\square\)
Lemma 3.3 (regularity for Gauss’s law). Let (A1)–(A7) be valid and let \( (E, \Phi, u, p, c) \in \mathbb{R}^{2+2n+L} \) be a solution of (2.1a)–(2.1j) according to Definition 3.1. Then, for the partial solution \( (E, \Phi) \), we have

\[
\Phi \in L^\infty(I; H^2(\Omega)/\mathbb{R}) \quad \text{and} \quad E \in L^\infty(I; H^1(\Omega)) .
\]

Proof. The proof is identical to the corresponding proof of Part 1 of this work. Furthermore, the proof holds even in three space dimensions.

4 Existence for multicomponent electrolytes

In this section, we prove that global weak solutions of the Darcy–Poisson–Nernst–Planck system exist.

4.1 A priori Estimates

We now show a priori bounds for the solution vector \( (E, \Phi, u, p, c) \in \mathbb{R}^{2+2n+L} \). First of all, we cite some preliminary results, which we use in the subsequent calculations.

Lemma 4.1 (Boundary Interpolation). Let \( u \in H^1(\Omega) \) and suppose (A1). Then, we have

\[
\|u\|_{L^2(\Gamma)}^2 \leq \delta \|\nabla u\|_{L^2(\Omega)}^2 + 2\delta^{-1} \|u\|_{L^2(\Omega)}^2 \quad \text{for all } \delta \in (0,1) .
\]

Proof. The stated inequality follows immediately from [1, Theorem 7.58], [1, Lemma 7.16], and Young’s inequality.

Lemma 4.2 (Approximation). Let \( I \subset \mathbb{R} \) be an interval and let \( u \in L^2(I; H^1(\Omega)) \cap H^1(I; H^1(\Omega)^*) \). Then, there exists a sequence \( (u_\varepsilon)_{\varepsilon>0} \subset C^1(I; H^1(\Omega)) \) such that we have

\[
\|u - u_\varepsilon\|_{L^2(I; H^1(\Omega))} + \|u - u_\varepsilon\|_{H^1(I; H^1(\Omega)^*)} + \|u - u_\varepsilon\|_{L^\infty(I; L^2(\Omega))} \to 0 .
\]

Proof. See [28, Lemma 7.2, Lemma 7.3].

Lemma 4.3 (Lyapunov function). Define the Lyapunov function \( \Lambda(x) : \mathbb{R}_+ \to \mathbb{R} \) by

\[
\Lambda(x) := x(\ln(x) - 1) + e \quad \implies \quad \Lambda'(x) = \ln(x) .
\]

Then, we have

\[
\Lambda(x) - x \geq 0 \quad \text{and} \quad \Lambda(x) \geq 0 .
\]

Proof. The second inequality follows with \( x \geq 0 \) from the first one. For the first inequality calculate the minimum of the function \( \Lambda(x) - x \). This shows \( \min_{x \geq 0}(\Lambda(x) - x) = 0 \).

Remark 4.4. We call the function \( \Lambda(x) \) a Lyapunov function, since for the heat equation \( \partial_t u - \Delta u = 0 \) in weak formulation, we easily obtain by formally testing with \( \ln(u) \) the estimate

\[
\|\Lambda(u(t))\|_{L^1(\Omega)} \leq \|\Lambda(u(0))\|_{L^1(\Omega)} .
\]

Hence, the function \( t \mapsto \|\Lambda(u(t))\|_{L^1(\Omega)} \) is non-increasing along every trajectory \( t \mapsto u(t) \). From the theory of ordinary differential equations, we know that functions with this property are called Lyapunov functions, cf. [4].

In the theory of partial differential equations, the Lyapunov function \( \Lambda(x) \) occurs naturally in estimates that are based on testing with the logarithm of the solution. These estimates are called entropy estimates, cf. [10], as in many situations, we can physically interpret for a solution \( u \) the logarithm \( \ln(u) \) as entropy, cf. [20]. For that reason, the estimates in Lemma 4.6 are called entropy estimates.

Lyapunov functions of the type \( \Lambda(x) \) have been used in [19] in the context of reaction-diffusion systems with reaction rates according to mass-action law.
Next, we repeat a short result from Part 1 of this work.

**Lemma 4.5** (Non negativity). Let (A1)–(A7) be valid and let \((E, \Phi, u, p, c) \in \mathbb{R}^{2+2n+L}\) be a weak solution of (2.1a)–(2.1j) according to Definition 3.1. Then, we have for \(l \in \{1, \ldots, L\}\)

\[ c_l(t, x) \geq 0 \quad \text{for a.e. } t \in [0, T_0], \text{ a.e. } x \in \Omega. \]

**Proof.** The proof is identical to the corresponding proof of Part 1 of this work. Furthermore, the proof holds even in three space dimensions. \(\square\)

We now prove the entropy estimates. Note, that these estimates continue to hold even for three space dimensions.

**Lemma 4.6** (Entropy estimates). Let (A1)–(A7) be valid and let \((E, \Phi, u, p, c) \in \mathbb{R}^{2+2n+L}\) be a solution of the system (2.1a)–(2.1j) according to Definition 3.1. Then, we have the estimates

\[
\sum_l \|\sqrt{c_l}\|_{L^2(I;L^\infty(\Omega))} + \|\nabla\sqrt{c_l}\|_{L^2(I;L^\infty(\Omega))} \leq C_L(T_0).
\]

**Proof.** Let \(\alpha > 0\). We test equations (3.1e) with \(\varphi := \ln(c_l + a) \in H^1(\Omega)\), we sum over \(l\) and we integrate in time over \([0, t_1]\) \(\in [0, T_0]\). For ease of readability, we split the following proof into two cases.

**Case 1:** \(c_l = \bar{c}_l\). For the sum of the time integrals, we obtain with Lemma 4.2 and Lemma 4.3

\[
\sum_l \left( \partial_t c_l, \ln(c_l + a) \right)_{L^2(0,t_1;H^1(\Omega))^*} \leq C_L(T_0)
\]

\[
= \sum_l \lim_{\varepsilon \to 0} \int_0^{t_1} \int_\Omega \partial_t (c_l + a) \ln(c_l + a) \, dx dt = \sum_l \sum_{\varepsilon = 0} \int_0^{t_1} \partial_A(c_l, + a) \, dx dt
\]

\[
= \sum_l \|A(c_l(t_1) + a)\|_{L^1(\Omega)} - \sum_l \|A(c_0, + a)\|_{L^1(\Omega)}.
\]

For the sum of the diffusion integrals, we immediately get with (A3)

\[
\sum_l \left( P \nabla c_l, \frac{1}{c_l + a} \nabla(c_l + a) \right)_{\Omega \times [0, t_1]} \geq \alpha_D \sum_l \|\nabla\sqrt{c_l + a}\|_{L^2(\Omega \times [0, t_1])}.
\]

The sum of the convection integrals, we transform with integration by parts and equation (3.1d) to

\[
I_{co} := -\sum_l \left( c_l u, \nabla \ln(c_l + a) \right)_{\Omega \times [0, t_1]}
\]

\[
= -\sum_l \left( (c_l + a)u, \nabla \ln(c_l + a) \right)_{\Omega \times [0, t_1]} - \sum_l \left( u, \nabla \ln(c_l + a) \right)_{\Omega \times [0, t_1]}
\]

\[
= -\sum_l \left( f, c_l + a \right)_{\Gamma \times [0, t_1]} - \sum_l \left( f, a \ln(c_l + a) \right)_{\Gamma \times [0, t_1]} =: A.1 + A.2.
\]

Applying Hölder’s inequality Lemma 4.1, and Lemma 4.3 leads for the integral \(I.1\) to

\[
A.1 \geq -\sum_l \|f\|_{L^\infty(\Gamma)} \|c_l + a\|_{L^1(\Gamma \times [0, t_1])} = -\sum_l \|f\|_{L^\infty(\Gamma)} \|\sqrt{c_l + a}\|_{L^2(\Gamma \times [0, t_1])}^2
\]

\[
\geq -\delta \sum_l \|\nabla \sqrt{c_l + a}\|_{L^2(\Omega \times [0, t_1])}^2 - 2\delta^{-1} \|f\|_{L^\infty(\Gamma)} \sum_l \|\sqrt{c_l + a}\|_{L^2(\Omega \times [0, t_1])}^2
\]

\[
= -\delta \sum_l \|\nabla \sqrt{c_l + a}\|_{L^2(\Omega \times [0, t_1])}^2 - 2\delta^{-1} \|f\|_{L^\infty(\Gamma)} \sum_l \|c_l + a\|_{L^1(\Omega \times [0, t_1])}^2
\]

\[
\geq -\delta \sum_l \|\nabla \sqrt{c_l + a}\|_{L^2(\Omega \times [0, t_1])}^2 - 2\delta^{-1} \|f\|_{L^\infty(\Gamma)} \sum_l \|\Lambda(c_l + a)\|_{L^1(\Omega \times [0, t_1])}^2.
\]
This yields for the sum of the convection integrals
\[
I_{co} \geq -\delta \sum_{i} \left\| \nabla c_i + a \right\|_{L^2(\Omega \times [0,t_i])}^2 - 2\delta^{-1} \left\| f \right\|_{L^\infty(\Gamma_T)}^2 \sum_{i} \left\| \Lambda(c_i + a) \right\|_{L^1(\Omega \times [0,t_i])} \\
- \sum_{i} \left( f, a \ln(c_i + a) \right)_{\Gamma \times [0,t_i]}.
\]

Similarly, we transform the sum of the electric drift integrals with integration by parts and equation (3.1b) to
\[
I_{de} := -\sum_{i} \left( c_i e z_i (e k_b T)^{-1} E, \nabla \ln(c_i + a) \right)_{\Omega \times [0,t_i]} \\
= -e (e k_b T)^{-1} \sum_{i} (E, z_i \nabla c_i)_{\Omega \times [0,t_i]} - a \sum_{i} e z_i (e k_b T)^{-1} (E, z_i \nabla \ln(c_i + a))_{\Omega \times [0,t_i]} \\
= e (e k_b T)^{-1} (\rho_f, \rho f)_{\Omega \times [0,t_i]} - e (e k_b T)^{-1} (\sigma, \rho f)_{\Gamma \times [0,t_i]} \\
+ \sum_{i} e z_i (e k_b T)^{-1} \left[ (\rho_f, a \ln(c_i + a))_{\Omega \times [0,t_i]} - (\sigma, a \ln(c_i + a))_{\Gamma \times [0,t_i]} \right] \\
=: B.1 + B.2 + B.3.
\]

Exactly as we treated the integral A.1, we come for the integral B.2 to
\[
B.2 = -e (e k_b T)^{-1} \sum_{i} (\sigma, z_i c_i)_{\Gamma \times [0,t_i]} \geq -e \max_i \left| z_i \right| (\sigma, c_i + a)_{\Gamma \times [0,t_i]} \\
\geq -\frac{e \max_i \left| z_i \right|}{e k_b T} \left\| \sigma \right\|_{L^\infty(\Gamma_T)} \left\| \sqrt{c_i + a} \right\|_{L^2(\Gamma \times [0,t_i])}^2 \\
\geq -\delta \sum_{i} \left\| \nabla c_i + a \right\|_{L^2(\Omega \times [0,t_i])}^2 - \frac{2e^2 \max_i \left| z_i \right|^2}{e k_b T^2} \left\| c_i + a \right\|_{L^1(\Omega \times [0,t_i])} \sum_{i} \left\| \Lambda(c_i + a) \right\|_{L^1(\Omega \times [0,t_i])}.
\]

Hence, we arrive for the sum of the electric drift integrals at
\[
I_{de} \geq e (e k_b T)^{-1} \left\| \rho_f \right\|_{L^2(\Omega \times [0,t_i])}^2 - \frac{2e^2 \max_i \left| z_i \right|^2}{e k_b T^2} \left\| \sigma \right\|_{L^\infty(\Gamma_T)}^2 \sum_{i} \left\| \Lambda(c_i + a) \right\|_{L^1(\Omega \times [0,t_i])} \\
- \delta \sum_{i} \left\| \nabla c_i + a \right\|_{L^2(\Omega \times [0,t_i])}^2 + \sum_{i} \left[ \rho_f, a \ln(c_i + a) \right]_{\Omega} - (\sigma, a \ln(c_i + a))_{\Gamma}.
\]

The remaining sum of the reaction integrals, we estimate with Lemma 4.3 and \(c_i \geq 0\) according to Lemma 4.5 by
\[
\sum_{i} \left( R_i(c_i), \ln(c_i + a) \right)_{\Omega \times [0,t_i]} \leq \max_i C R_i \sum_{i} \left( \frac{|c_i|}{(c_i + a)}, (c_i + a) \ln(c_i + a) \right)_{\Omega \times [0,t_i]} \\
\leq \max_i C R_i \sum_{i} \left( \frac{|c_i|}{(c_i + a)}, \Lambda(c_i + a) \right)_{\Omega \times [0,t_i]} \\
\leq \max_i C R_i \left( \frac{|c_i|}{\sum_{i} (c_i + a)}, \Lambda(c_k + a) \right)_{\Omega \times [0,t_i]} \leq \max_i C R_i \sum_{i} \left\| \Lambda(c_i + a) \right\|_{L^1(\Omega \times [0,t_i])}.
\]

Combining the preceding estimates leads with a suitable choice of the free parameter \(\delta > 0\) to the
Concerning the first stated estimate, we assume that

\[ \sum_l \| \Lambda(c_l(t_1) + a) \|_{L^1(\Omega)} + \frac{\alpha_D}{2} \sum_l \| \nabla \sqrt{c_l + a} \|_{L^2(\Omega \times [0, t_1])}^2 + \frac{e}{e_{k_b}T} \| \rho_f \|_{L^2(\Omega \times [0, t_1])}^2. \]

Finally, we define the constant \( a \). We now can safely let \( a \to 0 \). Thereby, we obtain with \( a \ln(c_l + a) \to 0 \) the entropy estimate

\[ \sum_l \| \Lambda(c_l(t_1)) \|_{L^1(\Omega)} + \frac{\alpha_D}{2} \sum_l \| \nabla \sqrt{c_l} \|_{L^2(\Omega \times [0, t_1])}^2 + \frac{e}{e_{k_b}T} \| \rho_f \|_{L^2(\Omega \times [0, t_1])}^2. \]

Applying Gronwall’s inequality leads immediately to

\[ \sum_l \| \Lambda(c_l) \|_{L^\infty(I, L^1(\Omega))} \leq [1 + bT_0 e^{\beta T_0}] \sum_l \| \Lambda(c_{0,l}) \|_{L^1(\Omega)} =: \bar{C}_L(T_0). \]

Hence, we obtain

\[ \| \rho_f \|_{L^2(\Omega \times [0, t_1])}^2 \leq \frac{e_{k_b}T}{e} \left[ \sum_l \| \Lambda(c_{0,l}) \|_{L^1(\Omega)} =: C_{L,1} \right]. \]

From this bound, we deduce with Lemma 4.3

\[ \sum_l \| \sqrt{c_l} \|_{L^\infty(I, L^2(\Omega))} + \frac{\alpha_D}{2} \sum_l \| \nabla \sqrt{c_l} \|_{L^2(\Omega \times [0, t_1])}^2 \leq \frac{2}{\alpha_D} \left[ \sum_l \| \Lambda(c_{0,l}) \|_{L^1(\Omega)} =: C_{L,2} \right]. \]

Finally, we define the constant \( C_L = C_L(T_0) \) by \( C_L := 2 \max(C_{L,1}, C_{L,2}) \).

**Case 2:** \( c_l \neq \tilde{c}_l \) Concerning the first stated estimate, we assume that \( \tilde{c}_l \) is contained in a ball in \( L^2(\Omega_T) \) with radius \( R \). Thus, we trivially obtain

\[ \| \tilde{\rho}_f \|_{L^2(\Omega_T)}^2 \leq \max_l \| \tilde{c}_l \|_{L^2(\Omega_T)} \leq LR^2 \max_l |z_l| . \]

Concerning the second stated estimate, we note that the bound of the time integrals, the diffusion integrals, the convection integrals, and the reaction integrals remain the same as in case 1 above. Only in the estimate for electric drift integral, the bound for the subintegral \( B.1 \) changes. More precisely, this time we get for the integral \( B.1 \) with 4.3

\[ B.1 \geq -e(ek_bT)^{-1} \max_l |z_l| \| \tilde{\rho}_f \|_{L^\infty(\Omega_T)} \sum_l \| c_l \|_{L^1(\Omega \times [0, t_1])} \]

\[ \geq -e(ek_bT)^{-1} \max_l |z_l| \| \tilde{\rho}_f \|_{L^\infty(\Omega_T)} \sum_l \| \Lambda(c_l + a) \|_{L^1(\Omega \times [0, t_1])} . \]
Analogously to case 1, we obtain after \(a \not\equiv 0\) and \(a \ln(c_i + a) \rightarrow 0\) the entropy estimate
\[
\sum_i \|\Lambda(c_i(t_1))\|_{L^2(\Omega)} + \frac{\alpha_D}{2} \sum_i \|\nabla c_i\|_{L^2(\Omega \times [0, t_1])}^2
\leq \sum_i \|\Lambda(c_0, t)\|_{L^1(\Omega)} + e(\epsilon_b T)^{-1} \max_i |z_i| \|\tilde{\rho}_f\|_{L^\infty(\Omega_T)} \sum_i \|\Lambda(c_i)\|_{L^1(\Omega \times [0, t_1])}
\]
\[
+ \frac{8}{\alpha_D} \int f \|\sigma\|_{L^\infty(\Gamma_T)}^2 \max_i \|\nabla c_i\|_{L^2(\Omega \times [0, t_1])}^2
\]
\[
\leq \frac{2}{\alpha_D} \left[ 1 + (b + b_0)C_L(T_0, R) \right] \sum_i \|\Lambda(c_0, t)\|_{L^1(\Omega)} =: C_L(T_0, R).
\]

Provided that \(c_i\) is contained in a ball in \(L^\infty(\Omega_T)\) with radius \(R\), we have with Gronwall’s inequality
\[
\sum_i \|\Lambda(c_i)\|_{L^\infty(\Omega_T)} \leq 1 + (b + b_0)T_0 e^{(b + b_0)T_0} \sum_i \|\Lambda(c_0, t)\|_{L^1(\Omega)} =: \tilde{C}_L(T_0, R).
\]

Thereby, we finally arrive at
\[
\sum_i \|\nabla c_i\|_{L^2(\Omega)}^2 + \frac{\alpha_D}{2} \sum_i \|\nabla c_i\|_{L^2(\Omega \times [0, t_1])}^2
\leq \frac{2}{\alpha_D} \left[ 1 + (b + b_0)C_L(T_0, R) \right] \sum_i \|\Lambda(c_0, t)\|_{L^1(\Omega)} =: C_L(T_0, R).
\]

Next, we prove the crucial energy estimates for the chemical species \(c_i\) with the aid of the entropy estimates from Lemma 4.6.

**Lemma 4.7 (Energy estimates).** Let (A1)–(A7) be valid and let \((E, \Phi, u, p, c) \in \mathbb{R}^{2+2n+L}\) be a weak solution of the system (2.1a)–(2.1j) according to Definition 3.1. Then, we have
\[
\sum_i \left[ \|c_i\|_{L^\infty(\Omega_T)} + \|c_i\|_{L^2(\Omega; L^2(\Omega))} \right] \leq C_0
\]
Herein, the dependency of the constant is
\[
C_0 = C_0 \left( C_L, T_0, \max_i |z_i|, \|f\|_{L^\infty(\Gamma_T)}, \|\sigma\|_{L^\infty(\Gamma_T)}, \|\tilde{\rho}_f\|_{L^\infty(\Omega_T)}, \|\tilde{c}_0, t\|_{L^2(\Omega)} \right).
\]

**Proof.** For ease of readability, we split the proof into two cases. Moreover, in the current proof we use Gagliardo-Nirenberg’s inequality in way, which restricts this result to two space dimensions.

**Case 1:** \(n = 2\) and \(c_i = \tilde{c}_i\). In equations (3.1e), we choose the test functions \(\varphi := c_i \in H^1(\Omega)\) and we sum over \(i = 1, 2\). Thereby, we get for the time integrals and the diffusion integrals with (A3)
\[
\sum_i \langle \partial_t c_i, c_i \rangle_{1, \Omega} + \sum_i \langle \nabla c_i, \nabla c_i \rangle_{\Omega} \geq \frac{\theta}{2} \frac{d}{dt} \sum_i \|c_i\|_{L^2(\Omega)}^2 + \alpha_D \sum_i \|\nabla c_i\|_{L^2(\Omega)}^2.
\]
For the convection integrals, we firstly use integration by parts and we insert equation (3.1d). Secondly, we use Hölder’s inequality and Lemma 4.1 with a rescaled free parameter \(\delta = \|f\|_{L^\infty(\Gamma_T)} \delta\). This leads us to
\[
- \sum_i \langle c_i u, \nabla c_i \rangle_{\Omega} = -\frac{1}{2} \sum_i \langle u, \nabla c_i \rangle_{\Omega} = -\frac{1}{2} \sum_i \langle f, c_i^2 \rangle_{\Gamma} \geq -\frac{1}{2} \sum_i \|f\|_{L^\infty(\Gamma_T)} \|c_i\|_{L^2(\Gamma)}^2
\]
\[
\geq -\delta \sum_i \|\nabla c_i\|_{L^2(\Omega)}^2 - \delta^{-1} \|f\|_{L^\infty(\Gamma_T)} \sum_i \|c_i\|_{L^2(\Omega)}^2.
\]
Analogously, for the electric drift integral we firstly integrate by parts, we insert equation (3.1b). Then, we use Hölder’s inequality and Lemma 4.1. This yields

\[ I_{el} := -e \sum_{l} (z_t c_t \mathbf{E}, \nabla c_t)_\Omega = \frac{e}{\varepsilon k_i T} \sum_{l} z_t \left( \rho_b + \tilde{\rho}_f, c_l^2 \right)_\Omega - \left( \sigma, c_l^2 \right)_\Gamma \]

\[ \geq \frac{e}{\varepsilon k_i T} \sum_{l} z_t \left( \rho_f, c_l^2 \right)_\Omega - \delta \sum_{l} |z_t| \|\nabla c_t\|_{L^2(\Omega)}^2 \]

\[ - \frac{2\varepsilon^2}{(\varepsilon k_i T)^2} \max_{l} |z_t| \left[ \delta^{-1} \|\sigma\|_{L^\infty(\Gamma_T)}^2 + \|\rho_b\|_{L^\infty(\Omega_T)} \right] \sum_{l} |z_t| \|c_l\|_{L^2(\Omega)}^2 \]

\[ := I.a + I.b + I.c. \]

We now have to bound the integral \( I.a \). For that purpose, we apply Hölder’s inequality, Gagliardo-Nirenberg’s inequality (we have \( n = 2 \)) and Young’s inequality, which yields

\[ I.a \leq \frac{e}{\varepsilon k_i T} \max_{l} |z_t| \|\rho_f\|_{L^2(\Omega)} \sum_{l} |c_l^2| \|\sigma\|_{L^2(\Omega)} \sum_{l} \|c_l\|_{L^2(\Omega)}^2 \]

\[ \leq \frac{e}{\varepsilon k_i T} \max_{l} |z_t| \|\rho_f\|_{L^2(\Omega)} \sum_{l} |c_l| \|\sigma\|_{L^2(\Omega)} \sum_{l} \|c_l\|_{L^2(\Omega)} \max_{l} |z_t| \|\rho_f\|_{L^2(\Omega)} \sum_{l} \|c_l\|_{L^2(\Omega)} \]

\[ \leq \delta \sum_{l} |c_l|_{H^1(\Omega)}^2 + \frac{e^2}{\delta (\varepsilon k_i T)^2} \max_{l} |z_t|^2 \|\rho_f\|_{L^2(\Omega)}^2 \sum_{l} |c_l|_{L^2(\Omega)}^2 . \]

Substituting the bound for \( I.a \), leads for electric drift integral to

\[ I_{el} \geq -\delta \sum_{l} \|\nabla c_t\|_{L^2(\Omega)}^2 - \frac{2\varepsilon^2}{(\varepsilon k_i T)^2} \left[ \|\sigma\|_{L^\infty(\Gamma_T)}^2 + \|\rho_b\|_{L^\infty(\Omega_T)} + \|\rho_f\|_{L^2(\Omega)}^2 \right] \sum_{l} |c_l|_{L^2(\Omega)}^2 . \]

For the reaction integrals, we applying (A5), Young’s inequality, and we recall \( c_l \geq 0 \). This results for the reaction integrals in

\[ \sum_{l} \left( \theta R_l(c), c_l \right)_\Omega \leq \theta \max_{l} C_{R_l} \left( |c|, \sum_{l} c_l \right)_\Omega \leq \theta \max_{l} C_{R_l} \left( \sum_{l} c_l, \sum_{l} c_l \right)_\Omega \]

\[ = \theta \max_{l} C_{R_l} \sum_{l} (c_l, c_l)_\Omega + \theta \max_{l} C_{R_l} \sum_{k \neq l} (c_k, c_l)_\Omega \leq 3 \theta C_{R_l} \sum_{l} |c_l|_{L^2(\Omega)}^2 . \]

By combining the preceding estimates, we deduce with the choice \( \delta := \frac{\alpha_D}{2} \) the estimate

\[ \frac{\theta}{2} \sum_{l} \|c_l\|_{L^2(\Omega)}^2 + \frac{\alpha_D}{2} \sum_{l} \|\nabla c_l\|_{L^2(\Omega)}^2 \]

\[ \leq \frac{12e^2 \max_{l} |z_t|^2}{\alpha_D (\varepsilon k_i T)^2} \left[ \|\sigma\|_{L^\infty(\Gamma_T)}^2 + \|\rho_b\|_{L^\infty(\Omega_T)} + \|\rho_f\|_{L^2(\Omega)}^2 \right] \sum_{l} |c_l|_{L^2(\Omega)}^2 \]

\[ + \left[ \frac{4}{\alpha_D} \||f\|_{L^2(\Gamma_T)}^2 + 3 \theta \max_{l} C_{R_l} \right] \sum_{l} |c_l|_{L^2(\Omega)}^2 . \]

With the abbreviations

\[ b(t) := \frac{12e^2 \max_{l} |z_t|^2}{\alpha_D (\varepsilon k_i T)^2} \|\rho_f\|_{L^2(\Omega)}^2 \]

\[ A_0 := \min \left( \frac{\theta}{2}, \frac{\alpha_D}{2} \right) \]

\[ B_0 := \frac{12e^2 \max_{l} |z_t|^2}{A_0 \alpha_D (\varepsilon k_i T)^2} \left[ \|\sigma\|_{L^\infty(\Gamma_T)}^2 + \|\rho_b\|_{L^\infty(\Omega_T)} + \|f\|_{L^2(\Gamma_T)}^2 + 3 \theta \max_{l} C_{R_l} \right] , \]
Again, we test equations (3.1e) with an a priori estimate supposed \( R_b \). Herein, we just changed the definition of the constant \( X \). We note, that in Definition 3.1 we introduced the space \( \| \cdot \|_{L^2(\Omega)} \). For the remaining part of the proof, we can shortly refer to Part 1 of this work, as we wrote Part 1 such that we can directly copy the following parts of the proof. The only difference is, that we subsequently deduce from the preceding estimate with Gronwall’s inequality and Lemma 4.6

\[
\sum_l \| c_l \|_{L^\infty(I; L^2(\Omega))}^2 \leq \exp \left( \int_0^{T_0} B_0 + b(t) \, dt \right) \sum_l \| \tilde{c}_0, l \|_{L^2(\Omega)}^2 \\
\leq e^{B_0 T_0} \exp \left( \frac{12 e^2 \max_l |z_l|^2}{\alpha_D (\epsilon_k T)^2} \| \tilde{\rho}_f \|_{L^2(\Omega_T)}^2 \right) \sum_l \| \tilde{c}_0, l \|_{L^2(\Omega)}^2 \\
\leq e^{B_0 T_0} \exp \left( \frac{12 e^2 \max_l |z_l|^2}{\alpha_D (\epsilon_k T)^2} C_L^2 \right) \sum_l \| \tilde{c}_0, l \|_{L^2(\Omega)}^2 := C_0^2 (T_0).
\]

Substituting this bound into the above estimate and integrating in time over \([0, T_0]\), yields the desired a priori estimate

\[
\sum_l \left[ \| c_l \|_{L^\infty(I; L^2(\Omega))} + \| \nabla c_l \|_{L^2(\Omega_T)} \right] \leq \tilde{C}_0 + B_0^{1/2} T_0^{1/2} \sum_l \| c_l \|_{L^\infty(I; L^2(\Omega))} \\
\leq \tilde{C}_0 + B_0^{1/2} T_0^{1/2} \tilde{C}_0 := C_0 (T_0).
\]

Case 2: \( n = 2 \) and \( c_l \neq \tilde{c}_l \) Again, we test equations (3.1e) with \( \varphi := c_l \in H^1(\Omega) \) and we run through the same calculations as already carried out in Case 1. The only difference is that we now immediately obtain for the integral \( I.a \)

\[
I.a \leq \frac{e}{\epsilon_k T} \max_l |z_l| \| \tilde{\rho}_f \|_{L^\infty(\Omega_T)} \sum_l \| c_l \|_{L^2(\Omega)}^2 \leq \frac{e}{\epsilon_k T} \max_l |z_l|^2 \| \tilde{e} \|_{L^\infty(\Omega)} \sum_l \| c_l \|_{L^2(\Omega)}^2
\]

We note, that in Definition 3.1 we introduced the space \( X \) and the set \( K \subset X \). Furthermore, we supposed \( e \in K \), which ensures that the \( L^\infty \)-norms of the \( \tilde{c}_l \) remain finite. Thus, provided we know \( \| \tilde{e} \|_{L^\infty(\Omega)} \leq R \) for all \( e \in K \), the constant in the above estimate just depends on an additional parameter \( R \). In conclusion, we analogously obtain

\[
\sum_l \left[ \| c_l \|_{L^\infty(I; L^2(\Omega))} + \| \nabla c_l \|_{L^2(\Omega_T)} \right] \leq \tilde{C}_0 + B_0^{1/2} T_0^{1/2} \tilde{C}_0 := C_0 (T_0, \| \tilde{e} \|_{L^\infty(\Omega)}).
\]

Herein, we just changed the definition of the constant \( b(t) \) by

\[
b(t) := \frac{12 e^2 \max_l |z_l|^2}{\alpha_D (\epsilon_k T)^2} \| \tilde{e} \|_{L^\infty(\Omega)} \leq \frac{12 e^2 \max_l |z_l|^2}{\alpha_D (\epsilon_k T)^2} \| \tilde{e} \|_{L^\infty(\Omega_T)} =: b_0.
\]

\[
\square
\]

4.2 Existence of a fixed point

For the remaining part of the proof, we can shortly refer to Part 1 of this work, as we wrote Part 1 such that we can directly copy the following parts of the proof. The only difference is, that we subsequently consider \( L \) solutes instead of two solutes. Thus, the range of the index \( l \) is now \( l = 1, \ldots, L \) instead of \( l = 1, 2 \).

**Lemma 4.8** (Boundedness). Let (A1)–(A7) be valid and let \((E, \Phi, u, p, c) \in \mathbb{R}^{2+2n+L}\) be a weak solution of the system (2.1a)–(2.1j) according to Definition 3.1. Then, we have

\[
\sum_l \| c_l \|_{L^\infty(\Gamma_T)} \leq C_M.
\]

Herein, the dependency of the constant is

\[
C_M = C_M \left( C_0, T_0, \max_l |z_l|, \| f \|_{L^\infty(\Gamma_T)}, \| \sigma \|_{L^\infty(\Gamma_T)}, \| \rho_r \|_{L^\infty(\Gamma_T)}, \| c_{0,l} \|_{L^\infty(\Omega)} \right).
\]
Proof. The proof is identical to the proof of the corresponding proof of Part 1 of this work.

Theorem 4.9 (A priori Bounds). Let (A1)–(A7) be valid and let \((E, \Phi, u, p, c) \in \mathbb{R}^{2+2n+L}\) be a weak solution of (2.1a)–(2.1j) according to Definition 2.2. Then, we have

\[
\|\Phi\|_{L^\infty(I;L^2(\Omega))} + \|E\|_{L^\infty(I;L^2(\Omega))} \leq C(T_0),
\]

\[
\|p\|_{L^\infty(I;L^2(\Omega))} + \|u\|_{L^\infty(I;L^2(\Omega))} \leq C(T_0),
\]

\[
\sum_i \left[\|c_i\|_{L^\infty(I;L^2(\Omega))} + \|c_i\|_{L^2(I;H^1(\Omega))} + \|c_i\|_{H^1(I;H^1(\Omega)^*)} + \|c_i\|_{L^\infty(\Omega_T)}\right] \leq C(T_0).
\]

Proof. The proof is identical to the corresponding proof of Part 1 of this work.

Theorem 4.10. Let (A1)–(A7) be valid. Then, there exists a solution \((E, \Phi, u, p, c) \in \mathbb{R}^{2+2n+L}\) of equations (2.1a)–(2.1j) according to Definition 2.2.

Proof. The proof is identical to the proof of the corresponding proof of Part 1 of this work.

5 Conclusion

In this paper, we showed the global existence of unique solutions of the Darcy–Poisson–Nernst–Planck system. The contribution of this paper was to deal with multicomponent electrolyte solutions, which consist of a neutral solvent and multiple charged solutes \(c_i\) with arbitrary valencies. In this situation, the main difficulty was to establish a priori estimates for the chemical species \(c_i\). By using entropy estimates, which have been developed for certain Lyapunov functionals, we successfully obtained such a priori estimates. However, these techniques are restricted to two space dimensions, as we combined the entropy estimates with Gagliardo-Nirenberg’s inequality in a way, which is valid only in two space dimensions.

In particular, by means of the entropy estimates the presented proof avoids further restrictions on the electrolyte solutions, such as the often used electroneutrality constraint or the volume additivity constraint. Therefore, our results can be applied to two dimensional models of general electrolyte solutions, which are captured by the Darcy–Poisson–Nernst–Planck system. This important especially in biological applications and hydrodynamical applications.

Finally, we note that if we restrict us to electrolyte solutions at rest (in this case the equations (2.1e)–(2.1g) coming from Darcy’s law vanish) and linear reaction rates, the presented model in this paper is identical to the considered model in [15], if additionally no constraints are involved in [15]. However, even in this situation we gave a new proof of the same result. More precisely, in [15] the crucial a priori estimates were obtained by involving a two dimensional, nonlinear version of Gagliardo-Nirenberg’s inequality from [7], whereas we used the entropy estimates for that purpose.

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