Global existence of weak solutions of a model for electrolyte solutions – Part 1: Two-component case

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Abstract

This paper analytically investigates the Darcy–Poisson–Nernst–Planck system. This system is a mathematical model for electrolyte solutions. In this paper, we consider electrolyte solutions, which consist of a neutral fluid and two suspended oppositely charged chemical species with arbitrary valencies \(z_1 > 0 > z_2\). We prove global existence and uniqueness of weak solutions in two space dimensions and three space dimensions.

So far, most of the existence results have been proven for symmetric electrolyte solutions. These solutions consist of a neutral fluid and two suspended charged chemical species with symmetric valencies \(\pm z\). As many electrolyte solutions in biological applications and hydrodynamical applications are not symmetric, the presented extension of the previous existence results is an important step.

Keywords: Global existence, electrolyte solution, electrohydrodynamics, Moser iteration, generalized Schauder fixed point theorem, Darcy–Poisson–Nernst–Planck system.

1 Introduction

Many complicated phenomena in hydrodynamics and biology can be modeled in the context of electrolyte solutions. The reason for this is that models for electrolyte solutions must simultaneously capture the following three ubiquitous processes: (i) the transport of the charged particles, (ii) the hydrodynamic fluid flow, (iii) the electrostatics. Moreover, these processes simultaneously occur in electrolyte solutions. Firstly, the electrostatic field is generated by the movement of the charged particles, and conversely, the movement of the charged particles is influenced by the electrostatic field. Secondly, the fluid flow changes the flux of the charged particles and conversely, the moving charged particles lead to a force term, which generates an electroosmotic fluid flow.

The classical models for electrolyte solutions, that capture the fully coupled nature of these processes are the so-called Poisson–Nernst–Planck systems (for a fluid at rest) and the Darcy–Poisson–Nernst–Planck systems (for laminar flow in porous media). In particular, Poisson–Nernst–Planck systems are also known as drift–diffusion systems, van Rosenbroeck systems, or semiconductor device equations. For a detailed derivation of these systems, we refer to [11, 12, 24, 28, 29, 33, 40, 41, 43]. Among many others, these models have been investigated analytically in [5, 6, 8, 9, 16, 17, 19, 22, 23, 27, 29, 37, 42–44].

So far, most of the analytical investigations have been carried out for electrolyte solutions, which consist of a electrically neutral solvent (at rest) and two oppositely charged chemical species with symmetric valencies \(\pm z\). One reason for this is that especially the symmetric valencies \(\pm 1\) naturally occur in the context of semiconductor devices and most of the analytical investigations are related to semiconductor devices. Previous existence results, which consider charged solutes with arbitrary valencies, were proven amongst others in [5, 8, 19, 37, 38]. These papers considered electrolyte solutions with multiple suspended charged solutes. We investigate this multicomponent case in the
second part of this work, whereas in this paper, we focus on the two-component case. This means, we consider electrolyte solutions, that consist of a neutral solvent and two oppositely charged solutes. In this situation, the results of this paper go beyond the above mentioned papers. More precisely, the authors of [5] proved local in time existence. The results of [37, 38] were proven under the additional assumption of a volume-additivity constraint and by including an additional reaction force term in the transport equations. These additional assumptions allow to bypass in [37, 38] the main difficulties, which we briefly sketch below. Finally, the paper [8] dealt with a stationary system and existence in two dimensions was established in [19].

In the proof of the crucial a priori estimates occur additional difficulties, if we allow for two oppositely charged solutes with arbitrary valencies \( z_1 > 0 > z_2 \). More precisely, the main difficulty is to obtain a priori estimates for the solutes \( c_l \), which are independent of the electric field. Such a priori estimates are easily obtained in case of symmetric valencies \( \pm z \). To briefly sketch this, we consider two charged solutes \( c_1 \) (positively charged) and \( c_2 \) (negatively charged) with symmetric valencies \( \pm z \). In the proof of a priori estimates for \( c_l \), we test the equations for \( c_l \) with the standard test functions \( \varphi = c_l \), and we remember that the electric field \( E \) satisfies according to Gauss’s law \( \nabla \cdot E = z(c_1 - c_2) \). Thereby, we obtain for the sum of the “electric drift integrals", which describe the electrophoretic motion

\[
-2 \sum_l \left( z (c_1 \varphi_l \cdot E)_{L^2(\Omega)} = z \left( \nabla \cdot (z(E \cdot (c_1)^2 - (c_2)^2))_{L^2(\Omega)} \right) \right.
\]

Due to this pointwise sign condition, we can omit the sum of the “electric drift integrals" and the a priori estimates for \( c_l \) are naturally independent of the electric field \( E \). In case of arbitrary valencies, such a pointwise sign condition does not hold true, if we use the standard test functions \( \varphi = c_l \). However, we propose to carry over this pointwise sign condition by using weighted test functions \( \varphi = |z| c_l \) instead.

The contribution of this paper is to prove global existence, uniqueness, and boundedness of weak solutions for electrolyte solutions, which consist of a neutral solvent and two charged solutes with arbitrary valencies. Moreover, we do not impose any further restrictions such as the often used electroneutrality constraint, cf. [3], or the volume additivity constraint, cf. [37]. This result is a first step towards the treatment of multicomponent electrolyte solutions, which contain \( L \in \mathbb{N} \) solutes. The presented proof is based on the weighted test functions \( \varphi = |z| c_l \). More precisely, we particularly use these weighted test functions in the proof of Lemma 5.4, which is the basis for the following a priori estimates in Lemma 5.6 and Theorem 5.7.

The rest of this paper is organized as follows: In Section 2, we present the Darcy–Poisson–Nernst–Planck system and in Section 3, we prove that solutions are unique. Then, we introduce the fixed point operator in Section 4. Finally, we show the crucial a priori estimates in Section 5.1, and in Section 5.2, we show the global existence.

## 2 Model Equations

Subsequently, we present the Darcy–Poisson–Nernst–Planck system. This system is a field-scale model\(^1\) for electrolyte solutions in porous media. A rigorous derivation of field-scale Darcy–Poisson–Nernst–Planck systems from pore-scale systems was carried out, e.g., in [3, 35]. Note that commonly on field-scales volume effects dominate and surface effects such as the electrostatic double-layer effects are negligible. However, a characteristic feature of porous media is are dominating surface effects even on field scales. This justifies to consider field-scale Darcy–Poisson–Nernst–Planck systems.

We now introduce some notation, in order to present the model equations.

\(^1\)For a detailed introduction to the modeling of porous media and the notion of field-scales and pore-scales, we refer to [4].
Simultaneously, the electric field carry charges, they interact with \( \Phi \) interplay between the electrophoretic movement of the charged particles, the electroosmotic flow of three conservation laws:

- For the electric field \( E \) and the initial data are obtained by substituting the initial data electrolyte solution, we have a background charge density \( \rho \) on \( \Omega \).
- Next, we denote by the velocity field \( \mathbf{u} \), its pressure \( p \), the electric field and the electrostatic potential in the electrolyte solution by \( E \) and \( \Phi \).
- The surface charges and the charged solutes give rise to an electric field \( E \).

We introduce \( \Omega_T := I \times \Omega \) a time space cylinder with lateral boundary \( \Gamma_T := I \times \partial \Omega \). Furthermore, we suppose \( \Omega \) to be a porous medium with constant porosity \( \theta \).

(N2) Variables: We assume that \( \Omega \) is fully saturated with a fluid, in which two charged chemical species are suspended. We denote the velocity field of the electrolyte solution by \( \mathbf{u} \), its pressure by \( p \), the electric field and the electrostatic potential in the electrolyte solution by \( E \) and \( \Phi \).

Next, we denote the number densities of the respective chemical species by \( c_l \), \( l \in \{1, 2\} \).

Furthermore, we define the concentration vector by \( c := (c_1, c_2) \).

(N3) Electrics: The chemical species \( c_l \) carry a charge \( e \z_l \). Here, \( e \) is the elementary charge and \( z_l \in \mathbb{Z} \) the respective valency \( (z_1 \neq 0) \). W.l.o.g., we assume \( z_1 > 0 > z_2 \). The chemical species \( c_l \) possess electric mobilities \( e \z_l \omega \), where \( \omega \) is the so-called mobility tensor. It is \( \mathcal{D} = k_b T \omega \) according to Einstein-Smoluchowski relation, see [28, Chapter 6]. Here, \( k_b \) is the Boltzmann constant, \( T \) the temperature. Hence, we have the identity \( e \z_1 \omega = e \z_2 (k_b T)^{-1} \mathcal{D} \). We denote by \( \rho_f \) the free charge density and by \( \rho_b \) a background charge density, e.g., coming from not resolved pore-scale inclusions inside \( \Omega \).

(N4) Coefficients: We denote by \( \mathcal{D} \) the diffusion-dispersion tensor, which is identical for all chemical species \( c_l \). Although the molecular diffusion might be different for each \( c_l \), the dispersion coming from a tortuous geometry is by far dominating on the considered field scales. Since the geometry looks the same for all chemical species, we obtain a coinciding diffusion-dispersion tensor \( \mathcal{D} \).

Next, we denote by \( \mathcal{K} \) the constant permeability tensor of the medium, by \( \mu \) the dynamic viscosity of the fluid, and by \( \mathcal{E} := \epsilon \mathcal{D} \) the constant electric permittivity tensor of the medium. For a rigorous derivation of the last relation, see [35]. We note, that we have the identity \( \mathcal{E} = \epsilon \mathcal{D} = k_b T \omega \).

We suppose the boundary \( \Gamma \) of the domain \( \Omega \) is charged, e.g., from surfactants. As the solutes \( c_l \) carry charges, they interact with \( \Gamma \) in a small boundary layer, the so-called electrostatic double-layer. This leads to a spatially inhomogeneous charge distributions, which gives rise to an electric field \( E \).

Simultaneously, the electric field \( E \) generates an electric body force in the surrounding fluid. Thereby, an electroosmotic flow develops, which in turn interacts with the chemical species. This leads to an interplay between the electrophoretic movement of the charged particles, the electroosmotic flow of the fluid, and a varying electric field.

Darcy–Poisson–Nernst–Planck systems capture these coupled processes based on the following three conservation laws:

**Law 1 – Gauss’s law:** The surface charges and the charged solutes \( c_l \) give rise to an electric field \( E \). For the electric field \( E \), we solve Gauss’s law. Additionally, we assume that the electric field is generated by an electrostatic potential \( \Phi \). Thus, we have \( E := -\nabla \Phi \). The boundary data are denoted by \( \sigma \) and the initial data are obtained by substituting the initial data \( c_{0,l} \) of the charged solutes \( c_l \) on the right hand side of Gauss’s law and solving for the electric field \( E \). Furthermore, we assume that inside the electrolyte solution, we have a background charge density \( \rho_b \), coming, e.g., from not resolved pore-scale inclusions inside \( \Omega \). Mathematically, Gauss’s law writes for the redefined electric field \( \mathcal{E} E \) as

\[
\mathcal{E}^{-1} E = -\nabla \Phi \quad \text{in } \Omega_T, \quad (2.1a)
\]
\[
\nabla \cdot E = \rho_f + \rho_b \quad \text{in } \Omega_T, \quad (2.1b)
\]
\[
\rho_f = \theta (z_1 c_1 - |z_2| c_2) \quad \text{in } \Omega_T, \quad (2.1c)
\]
\[
E \cdot \mathbf{u} = \sigma \quad \text{on } \Gamma_T, \quad (2.1d)
\]

**Law 2 – Darcy’s law:** The velocity field \( \mathbf{u} \) is subject to conservation of mass and momentum. On field-scales, this is sufficiently well-captured by Darcy’s law, which connects the velocity field \( \mathbf{u} \) and the pressure gradient \( \nabla p \). As we include electroosmotic flows, an electric body force term enters the equations. The boundary data are denoted by \( f \) and the initial data are obtained by inserting \( c_{0,l} \) and \( E(0) \) on the right hand side of Darcy’s law. Mathematically, Darcy’s law reads (with the redefined electric field \( E := \mathcal{E} E \)) as
In order to successfully examine the above model, we introduce the following assumptions.

We now introduce the required notation for the analytical investigations.

\[ \mathcal{K}^{-1} \mathbf{u} = \mu^{-1} \left( - \nabla p + \mathcal{E}^{-1} \rho_f \mathbf{E} \right) \quad \text{in } \Omega_T, \quad (2.1e) \]
\[ \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega_T, \quad (2.1f) \]
\[ \mathbf{u} \cdot \mathbf{n} = f \quad \text{on } \Gamma_T. \quad (2.1g) \]

**Law 3 – Nernst–Planck equations:** The evolution of the chemical species \( c_i \) is subject to mass continuity. Here, the mass flux arises due to diffusion, convection, and an electric drift. Such mass fluxes are called Nernst–Planck fluxes. We assume the equations for \( c_i \) are coupled through reaction rates \( R_i \). The initial data are denoted by \( c_{0,i} \) and the flux boundary data by \( g_i \). Mathematically, Nernst–Planck equations are given (with \( (N4) \) and the redefined electric field \( \mathbf{E} := \mathcal{E} \mathbf{E} \) by

\[ \theta \partial_t c_i + \nabla \cdot (D \nabla c_i + c_i (\mathbf{u} + e z_i (\varepsilon k T)^{-1} \mathbf{E})) = \theta R_i (c) \quad \text{in } \Omega_T, \quad (2.1h) \]
\[ (D \nabla c_i + c_i (\mathbf{u} + e z_i (\varepsilon k T)^{-1} \mathbf{E})) \cdot \mathbf{n} = g_i \quad \text{on } \Gamma_T, \quad (2.1i) \]
\[ c_i (0) = c_{0,i} \quad \text{on } \Omega. \quad (2.1j) \]

**Remark 2.1.** Equations (2.1a)–(2.1d) are Poisson’s equation for \( \Phi \) in mixed formulation. For this reason “Poisson” is contained in the name Darcy–Poisson–Nernst–Planck system. For analytical investigations, it is of advantage to deal with Poisson’s equation directly, as the comprehensive regularity results for Poisson’s equation hold true, cf. [18]. However, the mixed formulation is of advantage, as we can easily introduce in (2.1a) the general electric fields \( \mathbf{E} = - \nabla \Phi - \partial_t \mathbf{A} \), which is the expression of an electric field according to Maxwell’s equations in terms of the electromagnetic potentials, cf. [25]. Furthermore, the mixed formulation is of advantage as starting point for numerical approximations, since this leads to a direct approximation of the electric field \( \mathbf{E} \), cf. [14].

**Remark 2.2.** The boundary flux in equation (2.1i) can be equivalently expressed with equations (2.1d) and (2.1g) by

\[ (D \nabla c_i) \cdot \mathbf{n} = g_i - c_i f - (\varepsilon k T)^{-1} e z_i c_i \sigma \quad \text{on } \Gamma_T. \]

Thus, the boundary flux condition is equivalent to a Robin boundary condition for the diffusion part.

**Remark 2.3.** Equations (2.1a)–(2.1j) contain the nonlinear coupling terms \( \rho_f \mathbf{E} \) in Darcy’s law, and \( c_i \mathbf{u}, \mathbf{c_i E} \) in Nernst–Planck equations. These nonlinearities arise only after combing the three subprocesses to a Darcy–Poisson–Nernst–Planck system. This reflects the fact, that the coupling of initially isolated subprocesses leads to new nonlinearities in the resulting system.

### 2.1 Notation, Assumptions and Weak Formulation

We now introduce the required notation for the analytical investigations.

**(N5) Spaces:** For \( k > 0, p \in [1, \infty] \), we denote the Lebesgue spaces for scalar-valued and vector-valued functions by \( L^p (\Omega) \) and the respective Sobolev spaces by \( W^{k,p} (\Omega) \), cf. [2]. Furthermore, we set \( H^k (\Omega) := W^{k,2} (\Omega) \) and we refer for the definition of the Bochner spaces \( L^p (\mathcal{F}; V) \), \( H^k (\mathcal{F}; V) \) over a Banach space \( V \) to [39]. The \( H^1 (\mathcal{F}; \text{div}; \Omega) \)-spaces are defined, e.g., in [7] by \( H^1 (\mathcal{F}; \text{div}; \Omega) := \{ \mathbf{v} \in H^1 (\Omega) : \nabla \cdot \mathbf{v} \in L^2 (\Omega) \} \).

**(N6) Products:** We denote by \( \langle \cdot, \cdot \rangle_H \) the inner product on a Hilbert space \( H \) and by \( \langle \cdot, \cdot \rangle_{V \times V} \), the dual pairing between a Banach space \( V \) and its dual space \( V^* \). On \( \mathbb{R}^n \), we just write \( \mathbf{v} \cdot \mathbf{u} := \langle \mathbf{v}, \mathbf{u} \rangle_{\mathbb{R}^n} \) and on \( L^2 (\Omega) \), we just denote \( \langle \cdot, \cdot \rangle_{L^2 (\Omega)} := \langle \cdot, \cdot \rangle_{L^2 (\Omega)} \). In particular the dual pairing between \( H^1 (\Omega) \) and its dual \( H^1 (\Omega)^* \), we abbreviate by \( \langle \cdot, \cdot \rangle_{1,1} := \langle \cdot, \cdot \rangle_{H^1 (\Omega)^* \times H^1 (\Omega)} \).

In order to successfully examine the above model, we introduce the following assumptions

**(A1) Geometry:** Let \( n \in \{2, 3\} \) and \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain, i.e. \( \Gamma \in C^{0,1} \).

**(A2) Initial data:** The initial data \( c_{0,i} \) are non negative and bounded, i.e.,

\[ 0 \leq c_{0,i} (x) \leq M_0 \text{ for a.e. } x \in \Omega \text{ for some } M_0 \in \mathbb{R}_+. \]
(A3) **Ellipticity:** The diffusivity tensor $\mathcal{D}$ and the permeability tensor $\mathcal{K}$ satisfy
\[
\mathcal{D} \cdot \xi > \alpha_{D} |\xi|^{2} \quad \text{and} \quad \mathcal{K}^{-1} \cdot \xi > \alpha_{K} |\xi|^{2}
\]
for all $\xi \in \mathbb{R}^{n}$, $\mathcal{D} \cdot \eta < C_{D} |\eta|^{2}$ and $\mathcal{K}^{-1} \cdot \eta < C_{K} |\eta|^{2}$ for all $\xi, \eta \in \mathbb{R}^{n}$.

(A4) **Coefficients:** The porosity $\theta$, the dynamic viscosity $\mu$, and the electric permittivity $\epsilon$ are positive constants.

(A5) **Reaction rates:** The reaction rate functions $R_l : \mathbb{R} \to \mathbb{R}$ are global Lipschitz continuous functions, i.e., $R_l \in C^{0,1} (\mathbb{R}^{2})$ with Lipschitz constant $C_{R_l}$. Furthermore, we assume $R_l(0) = 0$ and $R_l(v) \geq 0$ for all $v \in \mathbb{R}$ with $v \leq 0$. This means, in case a chemical species vanishes, it can only be produced.

(A6) **Boundary data:** We assume $g_l \in L^{\infty} (\Omega_T)$, $l = 1, 2$, $\sigma \in L^{\infty} (\Omega_T)$, and $f \in L^{\infty} (\Omega_T)$.
Furthermore, we suppose that functions $f, \sigma \in L^{\infty} (I; W^{1,\infty} (\Omega))$ with $\sigma \cdot \nu = \sigma$ and $f \cdot \nu = f$ exist.

(A7) **Background charge density:** We assume $\rho_{b} \in L^{\infty} (\Omega_T)$ for the background charge density.

Equipped with the just introduced notation, we now define the weak formulation of the model.

**Definition 2.4 (Weak solution).** The vector $(E, \Phi, u, p, c) \in \mathbb{R}^{1+2n}$ is a weak solution of the Darcy–Poisson–Nernst–Planck system (2.1a)–(2.1j), if and only if

(i) $(E, \Phi) \in L^{\infty} (I; H_{s} (\text{div}; \Omega)) \times L^{\infty} (I; L^{2} (\Omega))$ solves for all $(v, \varphi) \in H_{0} (\text{div}; \Omega) \times L^{2} (\Omega)$
\[
\begin{align*}
(\mathcal{E}^{-1} E , v )_{\Omega} & = (\Phi , \nabla \cdot v )_{\Omega} \tag{2.2a} \\
(\nabla \cdot E , \varphi )_{\Omega} & = (\rho_{f} + \rho_{f} \cdot \varphi )_{\Omega} \tag{2.2b}
\end{align*}
\]
with the free charge density $\rho_{f}$ given by $\rho_{f} = \theta (z_{1} c_{1} - z_{2} c_{2})$.

(ii) $(u, p) \in L^{\infty} (I; H_{s} (\text{div}; \Omega)) \times L^{\infty} (I; L^{2} (\Omega))$ solves for all $(v, \varphi) \in H_{0} (\text{div}; \Omega) \times L^{2} (\Omega)$
\[
\begin{align*}
(\mathcal{K}^{-1} u , v )_{\Omega} & = (\mu^{-1} p , \nabla \cdot v )_{\Omega} + (\mu^{-1} \mathcal{E}^{-1} \rho_{f} E , v )_{\Omega} \tag{2.2c} \\
(\nabla \cdot u , \varphi )_{\Omega} & = 0 \tag{2.2d}
\end{align*}
\]

(iii) $c_{l} \in L^{\infty} (I; L^{2} (\Omega)) \cap L^{2} (I; H^{1} (\Omega)) \cap H^{1} (I; H^{1} (\Omega)^{*}) \cap L^{\infty} (\Omega_T)$ solves for all $\varphi \in H^{1} (\Omega)$ and for $l = 1, 2$
\[
\begin{align*}
\langle \partial_t c_{l} , \varphi \rangle_{1,\Omega} + (\mathcal{D} \nabla c_{l} , \nabla \varphi )_{\Omega} & = (c_{l} [u + e z_{l} (e k_{l} T)^{-1} E] , \nabla \varphi )_{\Omega} \tag{2.2e} \\
& = (\theta R_{l}(c) , \varphi )_{\Omega} + (g_{l} , \varphi )_{\Omega}
\end{align*}
\]
and $c_{l}$ takes its initial values in the sense that
\[
\lim_{t \to 0} (c_{l}(t) - c_{0,l} , \varphi )_{\Omega} = 0 \quad \text{for all } \varphi \in L^{2} (\Omega) .
\]

**Remark 2.5.** We note that equation (2.2e) is not well-defined without having $c_{l} \in L^{\infty} (\Omega_T)$. This is due to the fact, that an embedding of the type $H^{1} (\text{div}; \Omega) \hookrightarrow L^{p} (\Omega)$, for some $p > 2$, does not hold true. Thus, we have $E, u \in L^{2} (\Omega)$ at the best and for the existence of the convection integral and the electric drift integral in (2.2e), we need the estimate
\[
( c_{l} [u + e z_{l} (e k_{l} T)^{-1} E] , \nabla \varphi )_{\Omega} \leq \|c_{l}\|_{L^{\infty} (\Omega)} \|u + e z_{l} (e k_{l} T)^{-1} E\|_{L^{2} (\Omega)} \|\nabla \varphi\|_{L^{2} (\Omega)} .
\]
This shows that $c_{l} \in L^{\infty} (\Omega_T)$ is mandatory for a well-defined weak formulation. Consequently, we have to include $L^{\infty} (\Omega_T)$ in the solution space for $c_{l}$.

**Remark 2.6.** In equations (2.2a) and (2.2c), the test function space differs from the solution space and the solutions $E$ and $u$ are not admissible test functions. However, (2.1d), (2.1g), and (A6) ensure that $E - \sigma$ and $u - f$ are admissible test functions.
3 Uniqueness

In this section, we show that the solutions of the investigated Darcy–Poisson–Nernst–Planck system are unique.

**Theorem 3.1** (Uniqueness). Let (A1)–(A7) be valid and let \((E, \Phi, u, p, c) \in \mathbb{R}^{4+2n}\) be a weak solution of (2.1a)–(2.1j) according to Definition 2.4. Then, \((E, \Phi, u, p, c)\) is unique.

**Proof.** Let us assume that \((E_i, \Phi_i, u_i, p_i, c_i), i = 1, 2,\) are two solutions of (2.1a)–(2.1j) to identical data. Furthermore, we denote the difference between these two solution by

\[
(E_{12}, \Phi_{12}, u_{12}, p_{12}, c_{12}) := (E_1 - E_2, \Phi_1 - \Phi_2, u_1 - u_2, p_1 - p_2, c_1 - c_2)
\]

By subtracting the equations for the respective solutions, we obtain the error equations

**Gauss’s law:**

\[
(\nabla \cdot E_{12}, v)_{\Omega} = (\Phi_{12}, \nabla \cdot v)_{\Omega}, \tag{3.1a}
\]

\[
(\nabla \cdot E_{12}, \varphi)_{\Omega} = (\Phi_{12}, \varphi)_{\Omega} = \theta \sum_l (z_l c_{12}, \varphi)_{\Omega}. \tag{3.1b}
\]

**Darcy’s law:**

\[
(K^{-1} u_{12}, v)_{\Omega} = (\mu^{-1} p_{12}, \nabla \cdot v)_{\Omega} + \theta \mu^{-1} \sum_l (z_l c_{1,2} E^{-1} E_1 - z_l c_{1,2} E^{-1} E_2, v)_{\Omega} \tag{3.1c}
\]

\[
(\nabla \cdot u_{12}, \varphi)_{\Omega} = 0. \tag{3.1d}
\]

**Nernst–Planck equations:**

\[
(\theta \partial_t c_{12}, \varphi)_{1,\Omega} + (D \nabla c_{12}, \nabla \varphi)_{\Omega} - (c_l \mu^{-1} E_{12}, \nabla \varphi)_{\Omega}
+ (z_l c_{1,2} E^{-1} E_1 - z_l c_{1,2} E^{-1} E_2, \nabla \varphi)_{\Omega} = \theta (R_l(c_1) - R_l(c_2), \varphi)_{\Omega}. \tag{3.1e}
\]

We now show by contradiction that \(c_{1,12} \equiv 0\) for \(l = 1, 2\). To this end, we assume that

\[
\sum_l \| c_{1,12} \|^2_{L^2(\Omega)} > 0 \iff \exists \kappa > 0 \text{ such that } \sum_l \| c_{1,12} \|^2_{L^2(\Omega)} \geq \kappa. \tag{3.2}
\]

Next, we test equation (2.2e) with \(\varphi = c_{1,12}\) and we sum over \(l = 1, 2\). Thereby, we come for the time integral and the diffusion integral with (A3) to

\[
\sum_l \left( \theta \partial_t c_{12} \right)_{1,\Omega} + (D \nabla c_{12}, \nabla c_{12})_{\Omega} \geq \frac{\theta}{2} \frac{d}{dt} \sum_l \| c_{1,12} \|^2_{L^2(\Omega)} + \alpha D \sum_l \| \nabla c_{1,12} \|^2_{L^2(\Omega)}. \tag{3.3}
\]

For the reaction integral, we obtain with (A5)

\[
\theta \sum_l \left( R_l(c_1) - R_l(c_2) \right)_{\Omega} \leq \theta \sum_l C_{R_l} \| c_{1,12} \|_{L^1(\Omega)} \leq \theta \max C_{R_l} \sum_l \| c_{1,12} \|^2_{L^2(\Omega)}. \tag{3.4}
\]

The convection integral and the electric drift integral, we transform to

\[
- \sum_l \left( c_{1,1} u_{12} + z_l (e k_b T)^{-1} E_1 \right)_{\Omega} - c_{1,2} u_{12} + z_l (e k_b T)^{-1} E_2 \right)_{\Omega}, \nabla c_{12}\right)_{\Omega}
- \sum_l \left( c_{1,2} u_1 + z_l (e k_b T)^{-1} E_1 \right)_{\Omega} - \sum_l \left( c_{1,2} u_2 + z_l (e k_b T)^{-1} E_2 \right)_{\Omega}, \nabla c_{12}\right)_{\Omega} = A.1 + A.2,
\]

\[\]
and A.2, we estimate with (3.2) and Young’s inequality with a free parameter $\delta > 0$, cf. [18], by

$$A.2 \leq \delta \sum_{l} \| \nabla c_{l,12} \|^2_{L^2(\Omega)} + \frac{1}{\kappa} \frac{1}{2\delta} \| u_{2} + e_{2}(t) - 1 \| E_{12} \|^2_{L^2(\Omega)} \sum_{l} \| L_{12} \|^2_{L^\infty(\Omega)}$$

$$\leq \delta \sum_{l} \| \nabla c_{l,12} \|^2_{L^2(\Omega)} + \frac{1}{2\kappa\delta} \sum_{i} \left[ \| u_{i} + e_{2}(t) - 1 \| E_{1} \|^2_{L^2(\Omega)} \| c_{i} \|^2_{L^\infty(\Omega)} \right] \sum_{l} \| L_{12} \|^2_{L^2(\Omega)}.$$  

Analogously, we come for A.1 to

$$A.1 \leq \delta \sum_{l} \| \nabla c_{l,12} \|^2_{L^2(\Omega)} + \frac{2}{\kappa\delta} \sum_{i} \left[ \| u_{i} + e_{2}(t) - 1 \| E_{1} \|^2_{L^2(\Omega)} \| c_{i} \|^2_{L^\infty(\Omega)} \right] \sum_{l} \| L_{12} \|^2_{L^2(\Omega)}.$$  

Altogether, we arrive with a proper choice of a free parameter $\delta > 0$ at

$$\frac{\theta}{2} \frac{d}{dt} \sum_{l} \| c_{l,12} \|^2_{L^2(\Omega)} + \frac{\alpha_D}{2} \sum_{l} \| \nabla c_{l,12} \|^2_{L^2(\Omega)}$$

$$\leq \left( \theta \max_{l} C_{R_{i}} + \frac{4}{\kappa\alpha_D} \sum_{l} \left[ \| u_{i} + e_{2}(t) - 1 \| E_{1} \|^2_{L^2(\Omega)} \| c_{i} \|^2_{L^\infty(\Omega)} \right] \right) \sum_{l} \| c_{l,12} \|^2_{L^2(\Omega)}.$$  

We note that the initial values vanish due to $c_{l,12} = c_{l,1} - c_{l,1} = 0$. Thus, applying Gronwall’s inequality, cf. [13], yields

$$\sum_{l} \| c_{l,12}(t) \|^2_{L^2(\Omega)} \leq C \sum_{l} \| c_{l,12}(0) \|^2_{L^2(\Omega)} = 0 \quad \text{for a.e. } t \in [0, T_0].$$  

This is a contradiction to assumption (3.2). Hence, we have proven $c_{l,12} \equiv 0$.

We proceed by testing equation (3.1b) with $\varphi = \nabla \cdot E_{12}$. Thereby, we come with Young’s inequality and $c_{l,12} \equiv 0$ directly to

$$\| \nabla \cdot E_{12} \|^2_{L^2(\Omega)} \leq \theta \sum_{l} \| z_{l,i} \|^2_{L^2(\Omega)} = 0.$$  

Next, we test equation (3.1b) with $\varphi = \Phi_{12}$ and equation (3.1a) with $v = E_{12}$. By adding these equations, we get with (A3) and $c_{l,12} \equiv 0$

$$\varepsilon \alpha_D \| E_{12} \|^2_{L^2(\Omega)} \leq (\varepsilon^{-1} E_{12}, E_{12})_{\Omega} = \theta \sum_{l} (z_{l,12}, \Phi_{12})_{\Omega} = 0.$$  

We now test equation (3.1a) with $v \in H_{0}(\text{div}; \Omega)$, for which we assume that $\nabla \cdot v = \Phi_{12}$ and $\| v \|^2_{H^{1}(\text{div}; \Omega)} \leq C \| \Phi_{12} \|^2_{L^2(\Omega)}$ holds, cf. [34, Chapter 7.2]. This gives with (A3) and Young’s inequality

$$\| \Phi_{12} \|^2_{L^2(\Omega)} \leq \frac{2}{\varepsilon \alpha_D} \| E_{12} \|^2_{L^2(\Omega)} = 0.$$  

Hence, we have proven $\| \Phi_{12} \|^2_{L^2(\Omega)} + \| E_{12} \|^2_{H^{1}(\text{div}; \Omega)} \leq 0$, which means $\Phi_{12} \equiv 0$, $E_{12} \equiv 0$, $\nabla \cdot E_{12} \equiv 0$.

Analogously, we test equation (3.1d) with $\varphi = \nabla \cdot u_{12}$. This shows $\| \nabla \cdot u_{12} \|^2_{L^2(\Omega)} = 0$. Next, we test equation (3.1d) with $v = p_{12}$ and equation (3.1c) with $v = u_{12}$. Then, we add these equations and obtain with (A3), $c_{l,12} \equiv 0$, and $E_{12} \equiv 0$

$$\alpha_D \| u_{12} \|^2_{L^2(\Omega)} \leq \theta \mu^{-1} \sum_{l} \left( z_{l,12} E_{1} - z_{l,12} E_{1} - E_{12}, u_{12} \right) = 0.$$  

By testing equation (3.1c) with $v \in H_{0}(\text{div}; \Omega)$, for which we assume according to [34, Chapter 7.2] that $\nabla \cdot v = p_{12}$ and $\| v \|^2_{H^{1}(\text{div}; \Omega)} \leq C \| p_{12} \|^2_{L^2(\Omega)}$ holds, we come with (A3) and Young’s inequality to

$$\| p_{12} \|^2_{L^2(\Omega)} \leq \theta \mu^{-1} \sum_{l} \left( z_{l,12} E_{1} - z_{l,12} E_{1} - E_{12}, p_{12} \right) =: I.1 + I.2 + I.3.$$  

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We already know \( I.2 = 0 \) and \( I.3 \) is estimated with \( c_{l,12} \equiv 0 \) and \( E_{12} \equiv 0 \) by

\[
I.3 = \| \rho_{f,12} c^{-1} E_1 + \rho_{f,2} E_1 \|_{L^2(\Omega)}^2 = 0 .
\]

Thus, we have \( \| p_{12} \|_{L^2(\Omega)} = 0 \) and we have proven \( \| p_{12} \|_{L^2(\Omega)}^2 + \| u_{12} \|_{H^1(\Omega)}^2 \leq 0 \), which means \( p_{12} \equiv 0, u_{12} \equiv 0, \nabla \cdot u_{12} \equiv 0 \).

### 4 Fixed Point Operator

In the next sections, we prove the existence of solutions of the Darcy–Poisson–Nernst–Planck system by applying a fixed point approach. The idea behind this method of proof can be roughly summarized as follows:

Firstly, linearize the nonlinear system with a suitable linearization method. For that purpose, often well-known and widely used numerical linearization schemes are used. Concerning Darcy–Poisson–Nernst–Planck systems, the most famous linearization scheme for numerical computations is the so-called Gummel iteration, cf. [21].

Secondly, reformulate the linearized system by means of an abstract operator. This operator is exactly constructed such that the images of this operator are the solutions of the linearized system. Furthermore, the construction must by carried out in such a way, that the solutions of the nonlinear system are exactly the fixed points of this operator. Hence, the existence of solutions of the nonlinear system is equivalent to the existence of fixed points of the constructed operator.

Thirdly, it remains to prove that the operator satisfies the assumptions of a fixed point theorem, which allows to conclude that a fixed point exists. This is the reason why the most part of the subsequent proof consists in verifying the assumptions of the fixed point Theorem 5.10.

We now linearize the Darcy–Poisson–Nernst–Planck system by the following Gummel-type approach, which is sketched as follows.

(L.0) We replace the free charge density \( \rho_f \) by some given approximation \( \tilde{\rho}_f \).

(L.1) Thereby, we decouple Gauss’s law from the remaining Darcy–Poisson–Nernst–Planck system, as for a given \( \tilde{\rho}_f \), we obtain a solution \( (E, \Phi) \) of Gauss’s law independently of the remaining solution vector \((u, p, c)\). In the following Definition 4.1, this is formulated by means of the operator \( F_1 \).

(L.2) Next, we proceed by solving Darcy’s law. From (L.0) and (L.1) we know that we can take \((\tilde{\rho}_f, E, \Phi)\) as a given input. Hence, we obtain a solution \((u, p)\) independently of the remaining solution vector \((u, p)\). In the following Definition 4.1, this is formulated by means of the solution operator \( F_2 \).

(L.3) Finally, we know from (L.0)–(L.2), that we can treat \((\tilde{\rho}_f, E, \Phi, u, p)\) as a given input for the equations for \( c \), which gives immediately the remaining solution \( c \). This is formulated in the following Definition 4.1 by means of the solution operator \( F_3 \).

This Gummel-type linearization approach is rigorously formulated in the next definition.

**Definition 4.1 (Fixed point operator).** Let \( K \subset X \) be a subset of the Banach space \( X \), which is given by \( X := [L^\infty(I; L^2(\Omega))] \cap L^2(I; H^1(\Omega)) \cap H^1(I; H^1(\Omega)^*) \cap L^\infty(\Omega_T)]^2 \). We introduce the fixed point operator \( F \) by

\[
F := F_3 \circ F_2 \circ F_1 : K \subset X \to X .
\]

**Herein,** the suboperator \( F_1 \) is defined by

\[
F_1 : \begin{cases}
K \to X \times L^\infty(I; H_0(div; \Omega)) \times L^2(I; L^2(\Omega))/\mathbb{R} := Y \\
c \mapsto (c, E, \Phi), \text{ with } (E, \Phi) \text{ solving for all } (v, \phi) \in H_0(div; \Omega) \times L^2(\Omega)
\end{cases}
\]

\[
(E^{-1} E, v)_{\Omega} = (\Phi, \nabla \cdot v)_{\Omega}, \quad (\nabla \cdot E, \phi)_{\Omega} = (\rho_0 + \tilde{\rho}_f, \phi)_{\Omega},
\]

with the free charge density \( \tilde{\rho}_f \) given by \( \tilde{\rho}_f = \theta(z_1 c_1 - |z_2| c_2) \).
Furthermore, the suboperator \( F_2 \) is defined by

\[
F_2 : \begin{cases}
Y \rightarrow Y \times L^\infty( I ; H_f( \text{div}; \Omega) ) \times L^\infty( I ; L^2( \Omega ) / \mathbb{R} ) =: Z \\
(\xi, E, \Phi) \mapsto (\xi, E, \Phi, u, p) \text{, with } (u, p) \text{ solving for all } (v, \varphi) \in H_0(\text{div}; \Omega) \times L^2( \Omega )
\end{cases}
\]

\[
(K^{-1}u, \varphi)_\Omega = (\mu^{-1}p, \nabla \cdot v)_\Omega + (\mu^{-1}\epsilon^{-1}\Phi J E , v)_\Omega, \quad \nabla \cdot v , \varphi)_\Omega = 0.
\]

Finally, the suboperator \( F_3 \) is defined by

\[
F_3 : \begin{cases}
Z \rightarrow X \\
(\xi, E, \Phi, u, p) \mapsto c = (c_1, c_2) \text{, with } c_1 \text{ solving for all } \varphi \in H^1(\Omega) \text{ and } l = 1, 2
\end{cases}
\]

\[
(\theta \partial_t c_1, \varphi)_\Omega + (D \nabla c_1 , \nabla \varphi)_\Omega - (c_1 [u + c_2 (ekb^{-1})^{-1}E] , \nabla \varphi)_\Omega
\]

\[
= (\theta R_l(c), \varphi)_\Omega + (q_l, \varphi)_\Gamma,
\]

\[
\text{and } c_1 \text{ take its initial values in the sense that } \lim_{t \downarrow 0} (c_1(t) - c_0, \varphi)_\Omega = 0 \quad \text{for all } \varphi \in L^2(\Omega).
\]

Remark 4.2. We note that the fixed point operator \( F \) is solely a function of \( c \). For this reason, a fixed point of \( c \) of \( F \) is only a partial solution in the sense of Definition 2.4, as \( c \) only solves the equations (2.2e). However, the suboperators \( F_1, F_2, \) and \( F_3 \) contain the necessary information about the remaining partial solutions \( (\Phi, E) \) and \( (p, u) \). Furthermore, in case a fixed point \( \bar{c} = F(\bar{c}) \) exists, these supoperators ensure the existence of the partial solutions \( (\Phi, E) \) and \( (p, u) \) such that this yields the existence of a solution \( (E, \Phi, u, p, c) \in \mathbb{R}^{4+2n} \) according to Definition 2.4.

Lemma 4.3 (well-definedness). Let (A1)–(A7) be valid. Then, \( F : K \subset X \rightarrow X \) defined in Definition 4.1, is well-defined.

Proof. \( F_1 \) is well-defined in the first component, since \( F_1 \) is the identity in the first component. As to the components \( (E, \Phi) \), we know that for all \( c \in K \) unique solutions \( (E, \Phi) \in L^\infty(I;H_0(\text{div};\Omega)) \times L^\infty(I;L^2(\Omega)/\mathbb{R}) \) of (4.1a) and (4.1b) exist. This follows from [34, Theorem 7.4.1]. However, \( \Phi \) is only determined up to a constant. Imposing, e.g., a zero mean value constraint\(^2\), leads to uniqueness of \( \Phi \).

Furthermore, we note that the time variable \( t \) plays only the role of a parameter in the equations for \( (E, \Phi) \). This leads to uniform results with respect to \( t \). Hence, \( F_1 \) is well-defined.

\( F_2 \) is the identity in the first three components. For the last two components \( (u, p) \), we know that for all \( c \in K \times L^\infty(I;H_0(\text{div};\Omega)) \) unique solutions of (4.1c) and (4.1d) exist. This follows again from [34, Theorem 7.4.1]. Likewise, \( p \) is only determined up to a constant and we obtain uniqueness by imposing, e.g., a zero mean value constraint. The existence is uniform in time, as \( t \) plays just the role of a parameter. Thus, \( F_2 \) is well-defined.

Applying Rothe’s method, cf. [39, Chapter 8.2], [36], together with the regularities of \( E, \Phi, u, p \) (according to Theorem 5.7) guarantees with Lemma 5.3 and Lemma 5.6 the existence of unique weak solutions \( c \in X \) of equations (4.1e). Thus, \( F_3 \) is well-defined.

Lemma 4.4 (Regularity for Gauss’s law). Let (A1)–(A7) be valid and let \( (E, \Phi, u, p, c) \in \mathbb{R}^{4+2n} \) be a solution of (2.1a)–(2.1) according to Definition 4.1. Then, for the partial solution \( (E, \Phi) \), we have

\[
\Phi \in L^\infty(I;H^2(\Omega)/\mathbb{R}) \quad \text{and} \quad E \in L^\infty(I;H^1(\Omega)).
\]

Proof. We recall from [20, 34], that the equation

\[
(E \nabla \psi , \nabla \varphi)_\Omega = (\rho_b + \rho_f, \varphi)_\Omega + (\sigma, \varphi)_\Gamma \quad \text{for all } \varphi \in H^1(\Omega).
\]

\(^2\)The mean value of a function \( f \in L^1(\Omega) \) is defined by \( \frac{1}{|\Omega|} \int_{\Omega} f \, dx \).
possesses a unique solution $\psi \in H^2(\Omega)$, if we impose a zero mean value constraint\(^3\). As the time variable $t$ plays only the role of a parameter in the equation for $\psi$, we obtain all results uniformly in time. This yields $\psi \in L^\infty(I; H^2(\Omega))$. Hence, by defining

$$\Phi := \psi \in L^\infty(I; H^2(\Omega))$$

and

$$E := \mathcal{E} \nabla \psi \in L^\infty(I; H^1(\Omega)),$$

we obtain a solution of equations (4.1e) and (4.1b). Finally, we already know from the proof of Lemma 4.3, that the above constructed solution $(E, \Phi)$ is the unique solution of equations (4.1a) and (4.1b).

\[\square\]

5 Global Existence of a Solution

5.1 A priori Estimates

In this section, we show a priori bounds for the solution vector $(E, \Phi, u, p, c) \in \mathbb{R}^{4+2n}$. We begin with some preliminary results, which we need throughout the rest of this paper. Henceforth, we denote by $C$ a generic constant, which may change from line to line in the calculations.

Lemma 5.1 (Algebraic Inequality). Let $p \geq 0$ and $a, b \in \mathbb{R}$ with $a \geq 0$ and $b \geq 0$. Then, we have

$$(a - b)(a^p - b^p) = (b - a)(b^p - a^p) \geq 0.$$

Proof. The equality is obvious and the inequality follows by considering the cases $a \geq b$ and $a < b$. \[\square\]

Lemma 5.2 (Boundary Interpolation). Let $u \in H^1(\Omega)$ and suppose (A1). Then, we have

$$\|u\|^2_{L^2(\Gamma)} \leq \delta \|
abla u\|^2_{L^2(\Omega)} + 2\delta^{-1} \|u\|^2_{L^2(\Omega)}$$

for all $\delta \in (0, 1)$.

Proof. Let $s \in [1/2, 1)$ and $H^s(\Omega)$ be a fractional Sobolev space defined in [1, Chapter 7.35, 7.43]. According to [1, Theorem 7.58] and [1, Lemma 7.16], we have the embeddings

$$\|u\|_{L^2(\Gamma)} \leq C \|u\|_{H^s(\Omega)} \leq C \|u^s\|_{H^1(\Omega)} \|u\|^{1-s}_{L^2(\Omega)}$$

for $s \in [1/2, 1)$.

Then, we choose $s = 1/2$ and apply Young’s inequality with a free parameter $\delta \in (0, 1)$. \[\square\]

We now show a lower bound for the chemical species $c_l$.

Lemma 5.3 (Non-negativity). Let (A1)–(A7) be valid and let $(E, \Phi, u, p, c) \in \mathbb{R}^{4+2n}$ be a weak solution of (2.1a)–(2.1j) according to Definition 4.1. Then, we have for $l \in \{1, 2\}$

$$c_l(t, x) \geq 0 \quad \text{for a.e. } t \in [0, T_0], \quad \text{a.e. } x \in \Omega.$$

Proof. We note that the following proof is independent of $\tilde{c}_l \neq c_l$ or $\tilde{c}_l = c_l$. For $l = 1, 2$, we modify equations (4.1e) with $c_{l,+} := \max(c_l, 0)$ to

$$\begin{align*}
(\theta \partial_t c_l, \varphi)_{t,\Omega} + (D \nabla c_l, \nabla \varphi)_\Omega - (c_{l,+}[u + e z_l(c_k T)^{-1}E], \nabla \varphi)_\Omega \\
= (\theta R_l(c_l), \varphi)_{t,\Omega} + (g_l, \varphi)_\Gamma.
\end{align*}

\tag{5.1}

Obviously, equations (4.1e) and (5.1) are identical for nonnegative solutions $c_l$. This means that nonnegative solutions $c_l$ of (5.1) are solutions of (4.1e). Furthermore, by involving Theorem 3.1, we know that nonnegative solutions $c_l$ of (5.1) are the unique solutions of equations (4.1e). Hence, it suffices to show that (5.1) solely allows for nonnegative solutions.

\[\text{Footnote 2}\]

\[\text{Footnote 3}\]
To this end, we test (5.1) with $c_{l,-} := \min(c_l, 0)$. Thereby, we obtain for the time integral and the diffusion integral with (A3)
\[
\frac{\theta}{2} \sum_l \left(\frac{d}{dt} |c_l|^2 \right)_{L^2(\Omega)} + \frac{\alpha_D}{2} \sum_l \left(\frac{d}{dt} \|\nabla c_l\|^2_{L^2(\Omega)}\right) \geq \frac{\theta}{2} \sum_l \left(\frac{d}{dt} |c_l|^2 \right)_{L^2(\Omega)} + \frac{\alpha_D}{2} \sum_l \left(\frac{d}{dt} \|\nabla c_l\|^2_{L^2(\Omega)}\right).
\]

The convection integral and the electric drift integral vanish due to $\Omega \cap \{c_l < 0\} \cap \{c_l > 0\} = \emptyset$. For the reaction integrals and the surface integrals, we come with (A5), (A6), Lemma 5.2, and Hölder’s inequality to
\[
\frac{\theta}{2} \sum_l \left(\frac{d}{dt} |c_l|^2 \right)_{L^2(\Omega)} + \frac{\alpha_D}{2} \sum_l \left(\frac{d}{dt} \|\nabla c_l\|^2_{L^2(\Omega)}\right) \leq C \|g_l\|_{L^\infty(\Omega)} \sum_l \|c_l\|_{L^2(\Omega)}.
\]

Combining the previous estimates leads us to
\[
\frac{\theta}{2} \sum_l \left(\frac{d}{dt} |c_l|^2 \right)_{L^2(\Omega)} + \frac{\alpha_D}{2} \sum_l \left(\frac{d}{dt} \|\nabla c_l\|^2_{L^2(\Omega)}\right) \leq C \|g_l\|_{L^\infty(\Omega)} \sum_l \|c_l\|_{L^2(\Omega)}.
\]

It is either $\sum_l \|c_l\|_{L^2(\Omega)} = 0$ and we are done, or we have $\sum_l \|c_l\|_{L^2(\Omega)} \geq \kappa$, for some $\kappa > 0$. This gives
\[
\frac{\theta}{2} \sum_l \left(\frac{d}{dt} |c_l|^2 \right)_{L^2(\Omega)} + \frac{\alpha_D}{2} \sum_l \left(\frac{d}{dt} \|\nabla c_l\|^2_{L^2(\Omega)}\right) \leq \kappa^{-1} C \|g_l\|_{L^\infty(\Omega)} \sum_l \|c_l\|_{L^2(\Omega)}.
\]

Applying Gronwall’s inequality and (A2) immediately yields $\sum_l \|c_l\|_{L^2(\Omega)} = 0$. 

Next, we prove energy estimates for the chemical species $c_l$, by using the above mentioned weighted test functions, see Section 1. These energy estimates are crucial for all following results.

**Lemma 5.4 (Energy estimates).** Let (A1)–(A7) be valid and let $(\mathbf{E}, \Phi, \mathbf{u}, p, e) \in \mathbb{R}^{3+2n}$ be a weak solution of (2.1a)–(2.1) according to Definition 4.1. Then, we have
\[
\sum_\Gamma \left[\|c_l\|_{L^\infty(I; L^2(\Omega))} + \|c_l\|_{L^2(I; H^1(\Omega))}\right] \leq C_0
\]

Herein, the dependency of the constant is
\[
C_0 = C_0 \left(T_0, \max_{\Gamma} |z|, \|g_l\|_{L^\infty(\Omega)}, \|\mathbf{f}\|_{L^\infty(\Gamma)}, \|\mathbf{g}\|_{L^\infty(\Gamma)}, \|p_0\|_{L^\infty(\Omega)}, \|c_0\|_{L^2(\Omega)}\right).
\]

**Proof.** For ease of readability, we split the proof into two cases.

**Case 1:** $c_l = c_l$. In equations (4.1e), we choose the test functions $\varphi := |z| c_l \in H^1(\Omega)$ and we sum over $l = 1, 2$. Thereby, we get for the time integrals and the diffusion integrals with (A3)
\[
\sum_\Omega \left(\partial_t c_l, |z| c_l\right)_{L^2(\Omega)} + \sum_\Omega \left(D \nabla c_l, \nabla (|z| c_l)\right)_{L^2(\Omega)} \geq \frac{\theta}{2} \sum_\Omega \left(\frac{d}{dt} |c_l|^2 \right)_{L^2(\Omega)} + \frac{\alpha_D}{2} \sum_\Omega \left(\frac{d}{dt} \|\nabla c_l\|^2_{L^2(\Omega)}\right).
\]

For the convection integrals, we firstly use integration by parts and we insert equation (4.1d). Secondly, we use Hölder’s inequality and Lemma 5.2 with a rescaled free parameter $\delta = \|f\|_{L^\infty(\Gamma)} \delta$. This leads us to
\[
- \sum_\Omega \left(c_l u, \nabla (|z| c_l)\right)_{L^2(\Omega)} = -\frac{1}{2} \sum_\Omega \left|z\right| \left(\mathbf{u} \cdot \nabla c_l\right)_{L^2(\Omega)} = -\frac{1}{2} \sum_\Omega \left|z\right| \left(f, c_l^2\right)_{\Gamma}
\]
\[
\geq -\delta \sum_\Omega \left|z\right| \|\nabla c_l\|^2_{L^2(\Omega)} - \delta^{-1} \left\|f\right\|^2_{L^\infty(\Gamma)} \sum_\Omega \left|z\right| \|c_l\|^2_{L^2(\Omega)}.
\]
By combining the preceding estimates, we deduce with the choice of problems. See Section 1 for further details.

This sign condition yields

\[ \text{Analogously, for the electric drift integral we integrate by parts and we insert equation (4.1b). This yields} \]

\[ I_{el} := -\frac{e}{\epsilon k_b T} \sum_l (z_l c_l E, \, \nabla (|z_l| c_l))_\Omega = -\frac{e}{\epsilon k_b T} \sum_l \text{sign}(z_l) (|z_l| c_l E, \, \nabla (|z_l| c_l))_\Omega \]

\[ = \frac{e}{\epsilon k_b T} \sum_l \text{sign}(z_l) \left[ \left( \rho_b + \bar{\rho}_f \right, \, (|z_l| c_l)^2 \right)_\Omega - \left( \sigma, \, (|z_l| c_l)^2 \right)_\Gamma \right]. \]

Together with Hölder's inequality and Lemma 5.2, we reach from this identity (with a rescaled \( \delta \)) at

\[ I_{el} \geq \frac{e}{\epsilon k_b T} \sum_l \text{sign}(z_l) \left( \bar{\rho}_f, \, (|z_l| c_l)^2 \right)_\Omega - \delta \sum_l |z_l| \|\nabla c_l\|_{L^2(\Omega)}^2 \]

\[ \geq 2 e^2 \max_l |z_l|^2 \left[ \delta^{-1} \|\sigma\|_{L^\infty(\Gamma_T)}^2 + \|\rho_b\|_{L^\infty(\Omega_T)} \right] \sum_l |z_l| \|c_l\|_{L^2(\Omega)}^2 \]

\[ := I.a + I.b + I.c. \]

Now it remains to control the integral \( I.a \). As we assumed \( c_l = \bar{c}_l \), the free charge density \( \bar{\rho}_f \) is given by \( \bar{\rho}_f = z_1 c_1 - |z_2| c_2 \). This shows with Lemma 5.1

\[ I.a = \frac{e}{\epsilon k_b T} (z_1 c_1 - |z_2| c_2, \, (z_1 c_1)^2 - (|z_2| c_2)^2)_\Omega \geq 0. \]

This sign condition only holds true, since we use the weighted test function \( \varphi = |z_l| c_l \) instead of \( \varphi = c_l \). For the surface integrals, we involve (A6), Lemma 5.2, and Young's inequality. Thereby, we get

\[ \sum_l (g_l, \, |z_l| c_l)_\Omega \leq \sum_l |z_l| \|c_l\|_{L^2(\Gamma)} \sum_l |z_l| \|g_l\|_{L^2(\Gamma)} \]

\[ \leq \delta \sum_l |z_l| \|\nabla c_l\|_{L^2(\Omega)}^2 + 2 \delta^{-1} \sum_l |z_l| \|c_l\|_{L^2(\Omega)} \sum_l |g_l|_{L^2(\Gamma)}^2. \]

Applying (A5), Young's inequality, and recalling \( c_l \geq 0 \), results for the reaction integrals in

\[ \sum_l (\theta R_l(e), \, |z_l| c_l)_\Omega \leq \theta \max_l \left| \sum_l |z_l| c_l \right| \leq \theta \max_l \left( \sum_l |z_l| c_l, \, \sum_l |z_l| c_l \right)_\Omega \]

\[ \leq 3 \theta \max_l |z_l| c_l \sum_l \|z_l| c_l\|_{L^2(\Omega)}^2. \]

By combining the preceding estimates, we deduce with the choice \( \delta := \frac{\alpha_D}{\epsilon} \), the estimate

\[ \frac{\theta}{2} \frac{d}{dt} \sum_l |z_l| \|c_l\|_{L^2(\Omega)}^2 + \alpha_D \sum_l |z_l| \|\nabla c_l\|_{L^2(\Omega)}^2 \]

\[ \leq 2 e^2 \frac{\alpha_D}{(\epsilon k_b T)^2} \left[ \delta^{-1} \|\sigma\|_{L^\infty(\Gamma_T)}^2 + \|\rho_b\|_{L^\infty(\Omega_T)} \right] \sum_l |z_l| \|c_l\|_{L^2(\Omega)}^2 \]

\[ + \left[ \frac{6}{\alpha_D} \|f\|_{L^2(\Gamma_T)}^2 + 3 \theta \max_l |z_l| C_l \right] \sum_l |z_l| \|c_l\|_{L^2(\Omega)}^\infty + \max_l |z_l| \sum_l \|g_l\|_{L^2(\Gamma)}^2. \]  

(5.2)

For ease of readability, we introduce the abbreviation

\[ B_0 := \max \left( \frac{2}{\beta}, \, \frac{2}{\alpha_D} \right) \frac{12 e^2 \delta^{-1}}{\alpha_D (\epsilon k_b T)^2} \left[ \|\sigma\|_{L^\infty(\Gamma_T)}^2 + \|\rho_b\|_{L^\infty(\Omega_T)} + \|f\|_{L^2(\Gamma_T)}^2 + 3 \theta \max_l C_l \right]. \]

Thanks to \( \theta > 0 \), we can avoid to bound the integral \( I.a \) by suitable norms of its integrands, which would cause serious problems. See Section 1 for further details.
Thus, we immediately obtain from (5.2) together with Gronwall’s inequality
\[
\sum_{l} |z_l| ||c_l||^2_{L^\infty(I; L^2(\Omega))} \leq e^{B_0T_0} \left[ \sum_{l} |z_l| ||c_0,l||^2_{L^2(\Omega)} + \max_{l} |z_l| \sum_{l} \|g_l\|^2_{L^2(\Omega)} \right] := C_0^2(T_0).
\]

We substitute this bound into (5.2) and we integrate in time over \([0, T_0]\). This yields
\[
\sum_{l} |z_l| \left[ ||c_l||_{L^\infty(I; L^2(\Omega))} + \|\nabla c_l\|_{L^2(\Omega_T)} \right] \\
\leq \hat{C}_0 + B_0 \max_{l} |z_l| T_0 \hat{C}_0 + \max_{l} |z_l| \sum_{l} \|g_l\|_{L^2(\Omega)} := C_0(T_0) .
\]

(5.3)

\textbf{Case 2: } \textbf{c}_l \neq \textbf{c}_l. \quad \text{Again, we test equations (4.1e) with } \varphi := |z_l| c_l \in H^1(\Omega). \quad \text{The above estimates for the respective integrals remain unchanged, except for the integral } I.a, \text{ which does not fulfill a sign condition this time. Thus, we now bound this integral with Hölder’s inequality by}
\[
I.a = \frac{e}{e^{\mathbf{k}_cT}} \left( \rho f , \sum_{l} \text{sign}(z_l) \left( |z_l| c_l \right)^2 \right) _{\Omega} \leq \frac{e}{e^{\mathbf{k}_cT}} \max_{l} |z_l|^2 \|\mathbf{c}\|_{L^\infty(\Omega)} \sum_{l} \|z_l| ||c_l||^2_{L^2(\Omega)} .
\]

Herein, the constant depends on the \(L^\infty\)-norms of the \(c_l\). However, this is uncritical in this case as we assumed \(c_l \neq c_l\). Furthermore, in Definition 4.1 we introduced the space \(X\) and the set \(K \subset X\). Furthermore, we supposed \(c \in K\), which ensures that the \(L^\infty\)-norms of the \(c_l\) remain finite. Thus, provided we know \(|c|_{L^\infty(\Omega)} \leq R\) for all \(c \in K\), the constant in the above estimate just depends on an additional parameter \(R\). In conclusion, with the redefined constant
\[
B_0 := \frac{24e^2 \max_{l} |z_l|^2 C_{R}}{\min(\theta, \alpha_D) \alpha_D (\mathbf{k}_c T)^2} \left[ \|\mathbf{f}\|^2_{L^\infty(\Gamma_T)} + \|\rho\|_{L^\infty(\Gamma_T)} + \|f\|^2_{L^\infty(\Gamma_T)} + 3\theta + \|\mathbf{c}\|_{L^\infty(\Omega)} \right],
\]
we obtain analogously to (5.3)
\[
\sum_{l} |z_l| \left[ ||c_l||_{L^\infty(I; L^2(\Omega))} + \|\nabla c_l\|_{L^2(\Omega_T)} \right] \leq C_0(T_0, ||\mathbf{c}\|_{L^\infty(\Omega)}) .
\]

(5.4)

\textbf{Remark 5.5.} In particular, Lemma 5.4 ensures \( ||\rho f\|_{L^\infty(I; L^2(\Omega))} \leq \max_l |z_l| C_0\). This uniform bound holds for both cases, \( \rho f = \hat{\rho} f \) and \( \rho f \neq \hat{\rho} f \).

Next, we show that the chemical species \( c_l \) are bounded. As the proof is rather long and technical, we separate this proof from the proof of the remaining a priori bounds in Theorem 5.7.

\textbf{Lemma 5.6 (Boundedness). Let (A1)–(A7) be valid and let } \( (E, \Phi, u, p, c) \in \mathbb{R}^{4+2n}\) \textbf{be a weak solution of (2.1a)–(2.1l) according to Definition 4.1. Then, we have}
\[
\sum_{l} ||c_l||_{L^\infty(\Omega_T)} \leq C_M .
\]

Herein, the dependency of the constant is
\[
C_M = C_M \left( T_0, \max_l |z_l| , \|g_l\|_{L^\infty(\Omega_T)} , \|f\|_{L^\infty(\Gamma_T)} , \|\sigma\|_{L^\infty(\Gamma_T)} , \|\rho\|_{L^\infty(\Omega_T)} , \|c_0,l\|_{L^\infty(\Omega)} \right).
\]

\textbf{Proof.} As we have already established a lower bound for \( c_l \) in Lemma 5.3, it remains to show an upper bound. To this end, we subsequently apply Moser’s iteration technique, cf. [30, 31]. More precisely, we follow the proof of [26, Theorem 6.15] with a modified test function.

Henceforth, we use the truncated solutions \( c_l^n := \min(c_l, m) \). For ease of readability, we split the proof into several steps.
Step 1: preliminary energy estimates

The crucial step in Moser’s iteration technique is to derive an energy estimate for $c_l$ to arbitrary high powers, i.e., for $c_l^{\alpha+1}$ with $\alpha \geq 0$. For that purpose, we test equations (4.1e) by $\varphi := (c_l^{m})^{2\alpha+1}$ for $\alpha \geq 0$, we sum over $l = 1, 2$, and we bound the respective integrals. This part of the proof is related to the proof of Lemma 5.4. For this reason, we just briefly repeat the similar parts. Firstly, using the above test function yields with (A3) for the diffusion integrals

$$
(2\alpha + 1) \sum_l \left( D \nabla c_l , (c_l^{m})^{2\alpha} \nabla c_l^{m} \right) \geq \frac{\alpha \nu}{\alpha + 1} \sum_l \| \nabla (c_l^{m})^{\alpha+1} \|^2_{L^2(\Omega)} .
$$

The convection integrals, we firstly transform with integration by parts and inserting equation (4.1d).

Then, we involving Hölder’s inequality and Lemma 5.2 (with a rescaled parameter $\delta$). Thereby, we arrive for the convection integrals at to

$$
- \sum_l \left( c_l u , \nabla (c_l^{m})^{2\alpha+1} \right) \geq - \sum_l \frac{2\alpha + 1}{(2\alpha + 2)} \left( f , (c_l^{m})^{2\alpha+2} \right)_\Gamma.
$$

Analogously, we transform the electric drift integrals with equation (4.1b) to

$$
I_{el} := - \frac{e}{e_k} \sum_l \left( z_l \epsilon l c_l E , \nabla (c_l^{m})^{2\alpha+1} \right) \geq - \frac{e(2\alpha + 1)}{e_k T(2\alpha + 2)} \sum_l z_l \left( E , \nabla (c_l^{m})^{2\alpha+2} \right)_\Omega = - \frac{e(2\alpha + 1)}{e_k T(2\alpha + 2)} \sum_l z_l \left( \rho_b + \tilde{\rho}_f , (c_l^{m})^{2\alpha+2} \right)_\Gamma.
$$

Next, we apply Hölder’s inequality, and Lemma 5.2. Thereby, we come (with a rescaled $\delta$) to

$$
I_{el} \geq \frac{e(2\alpha + 1) \max_l |z_l|}{e_k T(2\alpha + 2)} \| \tilde{\rho}_f \|_{L^2(\Omega)} \sum_l \| (c_l^{m})^{2\alpha+2} \|_{L^2(\Omega)} - \delta \sum_l \| \nabla (c_l^{m})^{\alpha+1} \|^2_{L^2(\Omega)}
$$

$$
- \frac{2\epsilon^2}{(e_k T)^2} \max_l |z_l| \left[ \delta^{-1} \| \sigma \|_{L^\infty(\Gamma)} + \| \rho_b \|_{L^\infty(\Omega)} \right] \sum_l \| (c_l^{m})^{\alpha+1} \|^2_{L^2(\Omega)} =: K_0.
$$

We are done for integrals $I.b$ and $I.c$. As to the integral $I.a$, we apply Gagliardo-Nirenberg’s inequality, cf. [32], Young’s inequality, Lemma 5.4, and Remark 5.5. This shows

$$
I.a \geq \frac{e(2\alpha + 1)}{e_k T(2\alpha + 2)} \max_l |z_l| \| \tilde{\rho}_f \|_{L^2(\Omega)} \sum_l \| (c_l^{m})^{\alpha+1} \|^2_{L^2(\Omega)}
$$

$$
- \frac{e}{e_k T} \max_l |z_l|^2 C_0 C_{gn} \sum_l \| (c_l^{m})^{\alpha+1} \|_{L^2(\Omega)} \left( (\delta^{-1}) \right)^{n/2} \| (c_l^{m})^{\alpha+1} \|_{H^1(\Omega)}
$$

$$
\geq - \delta \sum_l \| \nabla (c_l^{m})^{\alpha+1} \|^2_{L^2(\Omega)} - C(\delta^{-1}) \frac{e^4}{(e_k T)^2} \max_l |z_l|^8 C_0 C_{gn}^4 \sum_l \| (c_l^{m})^{\alpha+1} \|^2_{L^2(\Omega)} =: K_i.
$$

Substituting this into the above intermediate estimate, results for electric drift integral in

$$
I_{el} \geq - \delta \sum_l \| \nabla (c_l^{m})^{\alpha+1} \|^2_{L^2(\Omega)} - [K_0 + K_i] \sum_l \| (c_l^{m})^{\alpha+1} \|^2_{L^2(\Omega)}.
$$
For the surface integrals, we involve (A6), Lemma 5.2. Furthermore, we apply Young’s inequality with \( q = \frac{2m+2}{m-1}, p = 2\alpha + 2 \). Thereby, we get
\[
\sum_i \left( g_i, \left(c_j^m\right)^{2\alpha+1} \right) \leq \sum_i \left\| \left(c_j^m\right)^{2\alpha+2} \right\|_{L^1(\Gamma)} + \sum_i \left\| g_i^{2\alpha+2} \right\|_{L^1(\Gamma)} \\
\leq \delta \sum_i \left\| \nabla \left(c_j^m\right)^{\alpha+1} \right\|_{L^2(\Omega)}^2 + 2\delta^{-1} \sum_i \left\| \left(c_j^m\right)^{\alpha+1} \right\|_{L^2(\Omega)}^2 + \sum_i \left\| g_i^{\alpha+1} \right\|_{L^2(\Gamma)}^2.
\]
A combination of the preceding estimates, together with the not yet considered time integrals and reaction integrals, yields with the choice \( \delta := \frac{\alpha_D}{\alpha D} \), the preliminary energy estimate
\[
\theta \sum_i \left\langle \partial_t c_i, \left(c_j^m\right)^{2\alpha+1} \right\rangle_{1,\Omega} + \frac{\alpha_D}{2(\alpha + 1)} \sum_i \left\| \nabla \left(c_j^m\right)^{\alpha+1} \right\|_{L^2(\Omega)}^2 \\
\leq \frac{12(\alpha + 1)}{\alpha D} \left[ \left\| f \right\|_{L^\infty(\Gamma_T)} + 1 + K_0 + K_1 \right] \sum_i \left\| \left(c_j^m\right)^{\alpha+1} \right\|_{L^2(\Omega)}^2 \\
+ \sum_i \left\| g_i^{\alpha+1} \right\|_{L^2(\Gamma)}^2 + \sum_i \left( \theta R_l(c), \left(c_j^m\right)^{2\alpha+1} \right)_{\Omega}.
\]

Step 2: base case: Before we continue the proof, we define for \( j \in \mathbb{N}_0 \), a sequence of exponents \( \alpha_j \) by
\[
1 + \alpha_j := \left( \frac{n + 2}{n} \right)^j \quad \Rightarrow \quad 1 + \alpha_j = (1 + \alpha_{j-1})(1 + \alpha_1).
\]
Additionally, we cite from [10, Proposition 3.2] the parabolic embedding
\[
\left\| c_i \right\|_{L^{\infty(m+2)}(\Omega_T)} \leq C_S(T_0, \Gamma, n) \left[ \left\| c_i \right\|_{L^\infty(I;L^2(\Omega))} + \left\| \nabla c_i \right\|_{L^2(\Omega_T)} \right].
\]
We now begin Moser’s iteration procedure, which we rigorously formulate as mathematical induction. We start with \( j = 0 \). Thus, we have \( 1 + \alpha_0 = 1 \), which means \( \alpha_0 = 0 \). Substituting this into (5.5), we can safely let \( m \to \infty \), as every integral remains finite. Furthermore, the reaction integrals and the time integrals, we estimate exactly as in the proof of Lemma 5.4. Thereby, we rediscover a slightly modified version of (5.2). I.e., we arrive with \( m \to \infty \) at
\[
\frac{\theta}{2} \sum_i \frac{d}{dt} \left\| c_i \right\|_{L^2(\Omega)}^2 + \frac{\alpha_D}{2} \sum_i \left\| \nabla c_i \right\|_{L^2(\Omega)}^2 \\
\leq \frac{12}{\alpha D} \left[ \left\| f \right\|_{L^\infty(\Gamma_T)} + 1 + K_0 + K_1 + 3\theta \max \left\| c_i \right\|_{L^2(\Omega_T)} \right] \sum_i \left\| c_i \right\|_{L^2(\Omega)}^2 + \sum_i \left\| g_i \right\|_{L^2(\Gamma)}^2.
\]
Again, we deduce with Gronwall’s inequality
\[
\sum_i \left[ \left\| (c_i) \right\|_{L^\infty(I;L^2(\Omega))} + \left\| \nabla (c_i) \right\|_{L^2(\Gamma_T)} \right] \leq \hat{C}_0 + B_0 \hat{T}_0^2 \hat{C}_0 + \sum_i \left\| g_i \right\|_{L^2(\Omega)}^2 := C_0(T_0).
\]
This time, we have denoted the constants by
\[
B_0 := \min \left( \frac{2}{\theta}, \frac{2}{\alpha D} \right) \frac{12 \max \left| z_i \right|}{\alpha D} \left[ \left\| f \right\|_{L^\infty(\Gamma_T)} + 1 + K_0 + K_1 + 3\theta \max \left\| c_i \right\|_{L^2(\Gamma_T)} \right],
\]
\[
\hat{C}_0 := \left( e^{B_0 T_0} \right) \left[ \sum_i \left\| c_{0,i} \right\|_{L^2(\Omega)}^2 + \sum_i \left\| g_i \right\|_{L^2(\Omega)}^2 \right].
\]
By combining (5.8a), the definition of \( \alpha_1 \) in (5.6), and the embedding (5.7), we finally obtain (with the elementary inequality \((a + b)^{1/p} \leq a^{1/p} + b^{1/p} \) for \( a, b \geq 0 \) and \( p \geq 1 \))

\[
\left[ \sum_l \|c_l\|_{L^{2(1+\alpha_1)}(\Omega_T)}^{2(1+\alpha_1)} \right]^{1/(1+\alpha_1)} \leq \sum_l \|c_l\|_{L^2(\Omega_T)} = \sum_l \|c_l\|_{L^{\infty}(\Omega_T)}^{2(1+\alpha_1)} \\
\leq C_S \sum_l \|c_l\|_{L^\infty(I; L^2(\Omega))} + \|\nabla c_l\|_{L^2(\Omega_T)} \leq C_SC_0 .
\]  

(5.8b)

**Step 3: induction hypothesis**  Let \( j \in \mathbb{N}_0 \) and let \( \alpha_j \) be the corresponding exponent defined in (5.6). We suppose, there exists a constant \( C_{j-1}(T_0) \) such that we have

\[
\sum_l \|c_l\|_{L^\infty(I; L^2(\Omega))} \leq C_{j-1}(T_0),
\]

(5.9a)

\[
\sum_l \|c_l\|_{L^2(\Omega_T)} \leq C_{j-1}(T_0) .
\]

(5.9b)

We note that (5.9a) and (5.9b) reduce for \( j = 0 \) exactly to (5.8a) and (5.8b).

**Step 4: inductive step:** We return with the choice of \( \alpha := \alpha_j \) to (5.5) and we can safely let \( m \to \infty \), as we already know (5.9b). For the reaction integrals, this leads analogously to the base case with Young’s inequality \((p = \frac{2\alpha_j+1}{2\alpha_j+2}, q = 2\alpha_j + 2)\) to

\[
\sum_l \left( \theta R_l(e), (c_l)^{2\alpha_j+1} \right)_\Omega \leq 3\theta \max_l C_Ri \sum_l \|c_l\|_{L^2(\Omega)}^{\alpha_j+1} .
\]

Additionally, we estimate the time integrals analogously to the base case. Thus, after letting \( m \to \infty \), by incorporating the bounds for the reaction integrals and the time integrals, and by introducing the abbreviations

\[
A_j := \min \left( \frac{\theta}{2\alpha_j+2}, \frac{\alpha_D}{2(\alpha_j+1)} \right),
\]

\[
B_j := \frac{12(\alpha_j+1)}{A_j \alpha_D} \left[ \|f\|_{L^\infty(\Gamma_T)} + 1 + K_0 + K_1 3\theta \max_l C_Ri \right] .
\]

we finally obtain the energy estimate

\[
\frac{d}{dt} \sum_l \|c_l\|_{L^2(\Omega)}^{\alpha_j+1} + \sum_l \|\nabla (c_l)^{\alpha_j+1}\|_{L^2(\Omega)}^2 \leq B_j \sum_l \|c_l\|_{L^2(\Omega)}^{\alpha_j+1} + \sum_l \|g_l^{\alpha_j+1}\|_{L^2(\Omega)}^2 .
\]

Hence, we conclude with Gronwall’s inequality the uniform bound

\[
\sum_l \|c_l\|_{L^\infty(I; L^2(\Omega))}^{2(\alpha_j+1)} \leq e^{B_j T_0} \left[ \sum_l \|c_0(l)\|_{L^2(\Omega)}^{\alpha_j+1} + \sum_l \|g_l^{\alpha_j+1}\|_{L^2(\Omega)}^2 \right] .
\]

Next, we integrate the above energy estimate in time over \([0, T_0]\) and we involve the preceding bound. Thereby, we deduce the stated inequality (5.9a)

\[
\sum_l \left[ \frac{d}{dt} \|c_l\|_{L^\infty(I; L^2(\Omega))} + \|\nabla (c_l)^{\alpha_j+1}\|_{L^2(\Omega_T)} \right] \\
\leq \tilde{C}_j^{\alpha_j+1} + B_j^{\frac{2}{2\alpha_j+2}} \tilde{C}_j^{\alpha_j+1} + \sum_l \|g_l^{\alpha_j+1}\|_{L^2(\Omega)} := C_j^{\alpha_j+1}(T_0) .
\]

(5.10a)
Furthermore, with the definition of $\alpha_j$ in (5.6), and the embedding (5.7), we arrive at the stated bound (5.9b)

$$
\left[ \sum_l \|c_l\|_{L^{2(1+\alpha_j+1)}(\Omega_T)}^{2(1+\alpha_j+1)} \right]^{\frac{1}{2(1+\alpha_j+1)}} \leq \left[ \sum_l \left( \|c_l\|_{L^{2(1+\alpha_j+1)}(\Omega_T)}^{\alpha_j+1} + \|\nabla (c_l)\|_{L^2(\Omega_T)} \right) \right]^{\frac{1}{\alpha_j+1}}.
$$

where we used

$$
\|c_l\|_{L^{2(1+\alpha_j+1)}(\Omega_T)} \leq C_S \sum_l \left( \|c_l\|_{L^{2(1+\alpha_j+1)}(\Omega_T)}^{\alpha_j+1} + \|\nabla (c_l)\|_{L^2(\Omega_T)} \right) \leq C_S C_j^{1+\alpha_j} \leq C_S^{\frac{1}{\alpha_j+1}} C_j.
$$

This shows that the induction hypothesis holds for all $j \in \mathbb{N}$.

**Step 5: limit case.** We now consider the limit case $j \to \infty$. First of all, we note that we have

$$
\|c\|_{L^p(\Omega)} := \left( \sum_l \|c_l\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty ,
$$

$$
\|c\|_{L^\infty(\Omega)} := \sum_l \|c_l\|_{L^\infty(\Omega_T)} , \quad \text{for } p = \infty .
$$

Hence, we can rewrite (5.10b) as

$$
\|c\|_{L^{2(1+\alpha_j+1)}(\Omega_T)} = \left[ \sum_l \|c_l\|_{L^{2(1+\alpha_j+1)}(\Omega_T)}^{2(1+\alpha_j+1)} \right]^{\frac{1}{2(1+\alpha_j+1)}} \leq C_S^{\frac{1}{\alpha_j+1}} C_j.
$$

Our goal is to show that this inequality holds even in the limit case $j = \infty$. For that purpose, we recall that $a^{1/p} \to 1$ as $p \to \infty$ for all $a > 0$. Furthermore, from the definition of $\alpha_j$ in (5.6), we know that $\alpha_j \to \infty$ as $j \to \infty$. Thus, with the definition of the constants $C_j$, we obtain in the limit

$$
C_\infty := \lim_{j \to \infty} \left( C_S^{\frac{1}{\alpha_j+1}} C_j \right) = \left( \lim_{j \to \infty} C_S^{\frac{1}{\alpha_j+1}} \right) \left( \lim_{j \to \infty} C_j \right) = \lim_{j \to \infty} C_j
$$

$$
\leq \lim_{j \to \infty} \hat{C}_j + \lim_{j \to \infty} B_j^{\frac{1}{1+\alpha_j+1}} \lim_{j \to \infty} T_0^{\frac{1}{1+\alpha_j+1}} \hat{C}_j + \lim_{j \to \infty} \sum_l \|g_l\|_{L^\infty(\Omega)}^{\frac{1}{\alpha_j+1}}
$$

$$
:= C_{\infty,1} + C_{\infty,2} + C_{\infty,3}.
$$

Next, we recall from [2, Theorem 2.14] that for all $u \in L^\infty(\Omega)$ holds

$$
\|u\|_{L^\infty(\Omega)} = \lim_{p \to \infty} \|u\|_{L^p(\Omega)} .
$$

This reveals immediately with (A6) that $C_{\infty,3} = \lim_{j \to \infty} \sum_l \|g_l\|_{L^{2(\alpha_j+1)}(\Omega_T)} = \sum_l \|g_l\|_{L^\infty(\Omega)}$. Furthermore, we arrive with (5.11), the definition of $\hat{C}_j$ and $(e^x)^y = e^{xy}$ for $x, y > 0$ at

$$
C_{\infty,1} \leq e^{\frac{B_0}{2T_0}} \left( \sum_l \|c_0,l\|_{L^\infty(\Omega)} + \sum_l \|g_l\|_{L^\infty(\Omega)} \right),
$$

where we used

$$
\lim_{j \to \infty} \exp\left( \frac{B_j}{2(1+\alpha_j)} \right) \leq \exp\left( \frac{B_0}{2} \right) \text{ due to the definition of } B_j \text{ and } B_0.
$$

Analogously, we get for $C_{\infty,2}$ with $j^{1/j} \to 1$ as $j \to \infty$

$$
C_{\infty,2} = \lim_{j \to \infty} B_j^{\frac{1}{1+\alpha_j}} T_0^{\frac{1}{1+\alpha_j}} \hat{C}_j \leq e^{\frac{B_0}{2T_0}} \left( \sum_l \|c_0,l\|_{L^\infty(\Omega)} + \sum_l \|g_l\|_{L^\infty(\Omega)} \right).
$$
Combining the preceding estimates, shows that
\[ C_\infty \leq 2e^{\frac{m_{\Omega_T}}{2}} \left( \sum_i \|c_{0,i}\|_{L^\infty(\Omega)} + \sum_i \|g_i\|_{L^\infty(\Omega)} \right) + \sum_i \|g_i\|_{L^\infty(\Omega)}. \]

We now can safely let \( j \to \infty \) in (5.10c). Thereby we finally arrive together with (5.11) at
\[ \sum_i \|c_i\|_{L^\infty(\Omega)} = \|c\|_{L^\infty(\Omega)} = \lim_{j \to \infty} \|c\|_{L^2(\Omega_T)} \]
\[ \leq 2e^{\frac{m_{\Omega_T}}{2}} \left( \sum_i \|c_{0,i}\|_{L^\infty(\Omega)} + \sum_i \|g_i\|_{L^\infty(\Omega)} \right) + \sum_i \|g_i\|_{L^\infty(\Omega)} =: C_M(T_0). \]

Finally, we show the desired a priori bounds for a solution of the Darcy–Poisson–Nernst–Planck system. These a priori bounds are crucial for the proof of Theorem 5.11.

**Theorem 5.7 (A priori Bounds).** Let (A1)–(A7) be valid and let \((E, \Phi, u, p, c) \in \mathbb{R}^{4+2n}\) be a weak solution of (2.1a)–(2.1) according to Definition 2.4. Then, we have
\[ \|\Phi\|_{L^\infty(I; L^2(\Omega))} + \|E\|_{L^\infty(I; L^2(\Omega))} \leq C(T_0), \]
\[ \|p\|_{L^\infty(I; L^2(\Omega))} + \|u\|_{L^\infty(I; L^2(\Omega))} \leq C(T_0), \]
\[ \sum_i \left[ \|c_i\|_{L^\infty(I; L^2(\Omega))} + \|c_i\|_{L^2(I; H^1(\Omega))} + \|c_i\|_{H^1(I; H^1(\Omega)^*)} + \|c_i\|_{L^\infty(\Omega_T)} \right] \leq C(T_0). \]

**Remark 5.8.** In the following proof, we derive the constants of the stated a priori estimates in detail. This reveals how the constants depend on the data. However, for the following Theorem 5.11, especially the dependency of the end time \( T_0 \) is of interest. This is the reason why we just stated \( C = C(T_0) \) for the constants.

**Proof.** For ease of readability, we split the proof of the stated a priori estimates into several steps.

**Step 1.1 – energy estimates and boundedness for \( c_i \):** These a priori bounds are shown in Lemma 5.4 and Lemma 5.6.

**Step 1.2 – estimates for \( \partial_t c_i \):** We abbreviate \( B := L^2(I; H^1(\Omega)) \). By involving equations (4.1e), we obtain the identity
\[ \theta \|\partial_t c_i\|_{L^2(I; H^1(\Omega)^*)} := \sup_{\|\varphi\|_{H^1} \leq 1} \left( \theta \|\partial_t c_i, \varphi\|_{B^*} \right) \]
\[ = \sup_{\|\varphi\|_{H^1} \leq 1} \left[ -(D\nabla c_i, \nabla \varphi)_{\Omega_T} + \left( \frac{c_i}{2} u + ez(e_k T)^{-1} E \right) \nabla \varphi \right]_{\Omega_T} + \left( \theta R(e), \varphi \right)_{\Omega_T} + \left( g, \varphi \right)_{B_T} \]
\[ =: I.1 + I.2 + I.3 + I.4. \]

For I.1 and I.3, we arrive with Hölder’s inequality, (A3), (A5), and Lemma 5.6 at
\[ I.1 + I.3 \leq C_D \|\nabla c_i\|_{L^2(\Omega_T)} \sup_{\|\varphi\|_{H^1} \leq 1} \|\nabla \varphi\|_{L^2(\Omega_T)} + C_R \sum_i \|c_{i}^l\|_{L^2(\Omega_T)} \sup_{\|\varphi\|_{H^1} \leq 1} \|\varphi\|_{L^2(\Omega_T)} \]
\[ \leq (C_D + C_R \sum_i) \|c_{i}^l\|_B \leq (C_D + \max C_{R_i}) C_0(T_0) =: C_1(T_0). \]

The integral I.2, we bound with Hölder’s inequality and Lemma 5.6 by
\[ I.2 \leq \left\| \frac{c_i}{2} u + ez(e_k T)^{-1} E \right\|_{L^2(\Omega_T)} \sup_{\|\varphi\|_{H^1} \leq 1} \|\nabla \varphi\|_{L^2(\Omega_T)} \]
\[ \leq \|c_{i}^l\|_{L^\infty(\Omega_T)} \left\| u + ez(e_k T)^{-1} E \right\|_{L^2(\Omega_T)} \leq C_M(T_0) \left\| u + ez(e_k T)^{-1} E \right\|_{L^2(\Omega_T)}. \]
For $I.A$, we immediately get with Hölder’s inequality and Lemma 5.2

$$I.A \leq \|g_i\|_{L^2(\Gamma_T)} \left[ \sup \|\varphi\|_{\mathcal{A}^1} \|\varphi\|_{L^2(\Gamma_T)} \right] \leq \|g_i\|_{L^2(\Gamma_T)} \left[ C \sup \|\varphi\|_{\mathcal{A}^1} \|\varphi\|_{L^2(\Gamma_T)} \right] \leq C \|g_i\|_{L^2(\Gamma_T)}.$$ 

Thus, by combining the estimates for $I.1 - I.4$, we have shown

$$\|\partial c_t\|_{L^2(I;H^1(\Omega))^*} \leq C_1(T_0) + C_M(T_0) \left\| u + e z_l (ek_b T)^{-1} E \right\|_{L^2(\Omega_z)} + C \|g_i\|_{L^2(\Gamma_T)}. \quad (5.12)$$

Step 1.3 – a priori estimates for $c_l$: We now put Lemma 5.4, Lemma 5.6, and the estimates (5.12) together. In anticipation of estimates (5.13) and (5.14), we obtain the desired a priori bound

$$\sum_l \left[ \|c_l\|_{L^\infty(I;L^2(\Omega))} + \|c_l\|_{L^2(I;H^1(\Omega))} + \|c_l\|_{H^1(I;H^1(\Omega))} + \|c_l\|_{L^\infty(\Omega)} \right] \leq C_0(T_0) + C_M(T_0) + C_1(T_0) + \max(1, e z_l (ek_b T)^{-1})(C_c + C_f) C_M(T_0) + C \|g_i\|_{L^2(\Gamma_T)}.$$

Step 2.1 – estimate for $\nabla \cdot E$: We test equation (4.1b) with $\varphi = \nabla \cdot E$. Thereby, we directly get with Young’s inequality

$$\|\nabla \cdot E\|_{L^2(\Omega)} \leq \|\rho_0\|_{L^2(\Omega)} + \theta \max_l |z_l| \sum_l \|\bar{c}_l\|_{L^2(\Omega)}.$$

Since this estimate holds uniformly in time, we take the supremum over $t \in [0, T_0]$ and come to

$$\|\nabla \cdot E\|_{L^\infty(I;L^2(\Omega))} \leq \|\rho_0\|_{L^\infty(I;L^2(\Omega))} + \theta \max_l |z_l| \sum_l \|\bar{c}_l\|_{L^\infty(I;L^2(\Omega))}.$$ 

Hence, together with Lemma 5.4 we finally have (assume $\|\bar{c}_l\|_{L^\infty(I;L^2(\Omega))} \leq R$ in case $c_l \neq \bar{c}_l$)

$$\|\nabla \cdot E\|_{L^\infty(I;L^2(\Omega))} \leq \left\{ \begin{array}{ll} \|\rho_0\|_{L^\infty(I;L^2(\Omega))} + \theta \max_l |z_l| R & \text{if } c_l \neq \bar{c}_l, \\
\|\rho_0\|_{L^\infty(I;L^2(\Omega))} + \theta \max_l |z_l| C_0 & \text{if } c_l = \bar{c}_l. \end{array} \right.$$ 

$$=: C_{L, c}(T_0, R).$$

Step 2.2 – estimate for $\Phi$: Next, we test equation (4.1a) with $v \in H_0(\text{div}; \Omega)$. Due to [34, Chapter 7.2], we can choose $v$ such that $\nabla \cdot v = \Phi$ and $\|v\|_{H^1(\text{div}; \Omega)} \leq K \|\Phi\|_{L^2(\Omega)}$ holds. This yields with (A3) and Young’s inequality

$$\|\Phi\|_{L^2(\Omega)} \leq \frac{K^2}{c_{\alpha_D}} \|E\|_{L^2(\Omega)} \quad \Rightarrow \quad \|\Phi\|_{L^\infty(I;L^2(\Omega))} \leq \frac{K^2}{c_{\alpha_D}} \|E\|_{L^\infty(I;L^2(\Omega))}.$$

Step 2.3 – estimate for $E$: Due to $E \in L^\infty(I;H_0(\text{div}; \Omega))$ and (A6), we test equation (4.1a) with $v = E - \sigma \in H_0(\text{div}; \Omega).$ In addition, we test equation (4.1b) with $\varphi = \Phi$. By adding these equations, we get with (A3) and Young’s inequality

$$\frac{1}{c_{\alpha_D}} \|E\|_{L^2(\Omega)} \geq (E^{-1} E, E)_\Omega \leq (E^{-1} E, \sigma)_\Omega - (\Phi, \nabla \cdot \sigma)_\Omega + (\rho_0 + \bar{\rho}_f, \Phi)_\Omega$$

$$\leq \frac{\delta_1}{c_{\alpha_D}} \|E\|_{L^2(\Omega)}^2 + \frac{1}{4 \delta_1} \|\sigma\|_{L^2(\Omega)}^2 + \delta_2 \|\Phi\|_{L^2(\Omega)}^2 + \frac{1}{4 \delta_2} \left[ \|\nabla \cdot \sigma\|_{L^2(\Omega)}^2 + \|\rho_0 + \bar{\rho}_f\|_{L^2(\Omega)}^2 \right].$$

Thus, we arrive with a suitable choice of $\delta_1, \delta_2$, the above estimate for $\Phi$, by taking the supremum over time, and with Lemma 5.6 at (we assume $c_l = \bar{c}_l$ and we skip the uncritical case $c_l \neq \bar{c}_l$)

$$\|E\|_{L^\infty(I;L^2(\Omega))} \leq \kappa \left[ \|\sigma\|_{L^\infty(I;H^1(\text{div}; \Omega))} + \|\rho_0\|_{L^\infty(I;L^2(\Omega))} + \theta \max_l |z_l| C_0 \right] := C_{2, c}(T_0).$$
Step 2.4 – a priori estimate for \( (E, \Phi) \): Collecting the preceding inequalities for \( \nabla \cdot E, E, \) and \( \Phi \) shows

\[
\| E \|_{L^\infty(I; H^1(\text{div}; \Omega))} + \| \Phi \|_{L^\infty(I; L^2(\Omega))} \leq C_{1,e} + C_{2,e} + \frac{K}{\sqrt{\epsilon D}} C_{2,e} =: C_e(T_0). \tag{5.13}
\]

Step 3.1 – estimate for \( \nabla \cdot u \): We test equation (4.1d) with \( \varphi = \nabla \cdot u \) and immediately obtain

\[
\| \nabla \cdot u \|_{L^2(\Omega)} = 0 \quad \text{and thus} \quad \| \nabla \cdot u \|_{L^\infty(I; L^2(\Omega))} = 0.
\]

Step 3.2 – estimate for \( p \): Next, we test equation (4.1c) with \( v = H_0(\text{div}; \Omega) \). According to [34, Chapter 7.2], we find a \( v \) such that \( \nabla \cdot v = p \) and \( \| v \|_{H^1(\text{div}; \Omega)} \leq K \| p \|_{L^2(\Omega)} \) holds. This leads us with (A3), Young’s inequality, Lemma 5.6, and (5.13) to (we assume \( c_1 = c_2 \) and we skip the uncritical case \( c_1 \neq c_2 \))

\[
\| p \|_{L^2(\Omega)}^2 \leq \delta K \| p \|_{L^2(\Omega)}^2 + \frac{\mu C_k}{2\delta} \| u \|_{L^2(\Omega)}^2 + \frac{1}{2\delta \epsilon D} \| \rho f E \|_{L^2(\Omega)}^2
\]

\[
\leq \delta K \| p \|_{L^2(\Omega)}^2 + \frac{\mu C_k}{2\delta} \| u \|_{L^2(\Omega)}^2 + \frac{\theta \max |z_i|}{2\delta \epsilon D} C_c^2 C_M^2.
\]

A suitable choice of \( \delta > 0 \) immediately shows

\[
\| p \|_{L^\infty(I; L^2(\Omega))} \leq 2\mu K C_k \| u \|_{L^\infty(I; L^2(\Omega))} + \frac{2\mu K \theta \max |z_i|}{\epsilon_D} C_c(T_0) C_M(T_0) =: C_{1,f}.
\]

Step 3.3 – estimate for \( u \): We test equation (4.1d) with \( \varphi = p \) and equation (3.1c) with the test function \( v = u - f \). Here, we take \( f \) according to (A6), which ensures \( v \in H_0(\text{div}; \Omega) \). Furthermore, adding these equations, yields with (A3) and Young’s inequality, Lemma 5.6, and (5.13) (again, we assume \( c_1 = c_2 \) and we skip the uncritical case \( c_1 \neq c_2 \))

\[
\alpha K \| u \|_{L^2(\Omega)}^2 \leq - \left( \mu^{-1} p, \nabla \cdot f \right)_\Omega + (K^{-1} u, f)_\Omega + (\mu^{-1} \mathcal{E}^{-1} \rho f E, u - f)_\Omega
\]

\[
= \delta_1 \left[ \| \nabla \cdot f \|_{L^2(\Omega)}^2 + \frac{1}{2\mu^2 \delta_1} \| \nabla \cdot f \|_{L^2(\Omega)}^2 + \frac{1}{2\mu^2 \delta_1} \| \nabla \cdot f \|_{L^2(\Omega)}^2 + \theta \max_i |z_i| \right]
\]

\[
\leq \delta_1 \left[ \| \nabla \cdot f \|_{L^2(\Omega)}^2 + \frac{1}{2\mu^2 \delta_1} \| \nabla \cdot f \|_{L^2(\Omega)}^2 + \frac{1}{2\mu^2 \delta_1} \| \nabla \cdot f \|_{L^2(\Omega)}^2 + \theta \max_i |z_i| \right] C_c^2 C_M^2.
\]

We now insert the estimate for \( p \) and we choose \( \delta_1 \) and \( \delta_2 \) appropriately. Thereby, we directly arrive with taking the supremum over time at

\[
\| u \|_{L^\infty(I; L^2(\Omega))} \leq \left( \frac{4 K C_k}{\mu \alpha K} + \frac{2 C_k}{\alpha K} + \frac{2}{\alpha K} \right) \| f \|_{H^1(\text{div}; \Omega)}^2
\]

\[
+ \theta \max_i |z_i| \left[ \frac{8 K}{\epsilon_D \alpha K} + \frac{1}{\epsilon_D \alpha K} \right] C_c C_M =: C_{2,f}(T_0).
\]

Step 3.4 – a priori estimate for \( (u, p) \): Combing the preceding estimates for \( \nabla \cdot u, u, p \) shows

\[
\| u \|_{L^\infty(I; H^1(\text{div}; \Omega))} + \| p \|_{L^\infty(I; L^2(\Omega))} \leq C_{2,f} + 2 \mu K C_k C_{2,f} + C_{1,f} =: C_f(T_0). \tag{5.14}
\]

\[\square\]

Remark 5.9. The proof of Theorem 5.7 is valid in arbitrary space dimensions, i.e., for \( \Omega \subset \mathbb{R}^n \) with \( n \geq 2 \). However, in (A1) we restrict ourselves to \( n \leq 3 \), as we use in the proof of Theorem 5.11 compact embeddings of Aubin-Lions-type, which are valid only for \( n \leq 3 \). \[\square\]
5.2 Existence of a fixed point

In this section, we prove the existence of global weak solutions of the Darcy–Poisson–Nernst–Planck system. Our proof is based on the following fixed point theorem, see [45, Corollary 9.6].

**Theorem 5.10.** Let $F : K \subset X \to K$ be continuous, where $K$ is a nonempty, compact, and convex set in a locally convex space $X$. Then, $F$ has a fixed point.

A Banach space $X$ equipped with the weak* topology is a locally convex space $(X, \text{weak}^*)$. Hence, the above fixed point theorem is tailored for Banach spaces, which carry the weak* topology. In our case, the weak* topology is the natural choice for the following three reasons:

Firstly, the a priori estimates from Section 5.1 are equivalent to weak* compactness. Secondly, the solution space for $c_1$ includes $L^\infty(\Omega_T)$, which is not reflexive. Hence, the weak* topology differs from the weak topology. Thirdly, when using the weak* topology, we can reuse the a priori estimates from Section 5.1 for the weak* contnuitiy of the fixed point operator.

In summary, we can exaggeratedly state that in the weak* topology, the compactness of $K$ and the continuity of $F$ is already contained in the a priori estimates. However, this is valid, only if the predual of the solution space is separable. In this case, the set-based topological and the sequences-based ones coincide. This enables us to prove the continuity of the operator with weak* convergent sequences, instead of investigating preimages of weak* open sets.

**Theorem 5.11.** Let (A1)–(A7) be valid. Then, there exists a solution $(E, \Phi, u, p, c) \in \mathbb{R}^{4+2n}$ of equations (2.1a)-(2.1j) according to Definition 2.4.

**Proof.** For ease of readability, we split the proof into several steps

**Step 1 – the space $X$:** First of all, we repeat the definition of the space $X$ from Definition 4.1

$$X := \left[ L^\infty(I; L^2(\Omega)) \cap L^2(I; H^1(\Omega)) \cap H^1(I; H^1(\Omega)^*) \cap L^\infty(\Omega_T) \right]^2.$$

Furthermore, we equip $X$ with the norm

$$\|\cdot\|_X := \|\cdot\|_{L^\infty(I; L^2(\Omega))} + \|\cdot\|_{L^\infty(I; H^1(\Omega))} + \|\cdot\|_{H^1(I; H^1(\Omega))^*} + \|\cdot\|_{L^\infty(\Omega_T)}.$$

Thus, $(X, \|\cdot\|_X)$ is a Banach space. However, we henceforth consider the locally convex space $(X, \text{weak}^*)$ and all topological terms refer to the weak* topology. Furthermore, the predual $X_0$ of $X$ can be written according to [15, Chapter I, IV]) as

$$X_0 := \left[ L^1(I; L^2(\Omega)) \cap L^2(I; H^1(\Omega)^*) \cap H^1(I; H^1(\Omega))^* \cap L^1(\Omega_T) \right]^2.$$

Hence, $X_0$ is a separable Banach space with dual $X$ and the topological terms for $(X, \text{weak}^*)$ based on sets are equivalent with those based on sequences, cf. [15, 45]. In particular, the notion of weak* continuous/compact is equal to sequentially weak* continuous/compact.

**Step 2 – the set $K$:** For $R > 0$, we introduce the set $K$ as a ball of radius $R$ in $X$, i.e.,

$$K := \{ v = (v_1,v_2) \in X : \|v\|_X \leq R \} \subset X.$$

$K$ is nonempty, convex, and weak* compact due to Banach-Alaoglu-Bourbaki theorem, cf. [39, Theorem 1.7].

**Step 3 – the operator $F$:** We consider the operator $F$, which was already introduced in Definition 4.1. This operator is a well-defined operator due to Lemma 4.3.

**Step 4 – self mapping property $F(K) \subset K$:** Let $c \in K$. The definition of the set $K$ and the definition of the norm $\|\cdot\|_X$ ensure that we have for $t = 1,2$

$$\|\tilde{c}_t\|_{L^\infty(I; L^2(\Omega))} + \|\tilde{c}_t\|_{L^\infty(I; H^1(\Omega))} + \|\tilde{c}_t\|_{H^1(I; H^1(\Omega)^*)} + \|\tilde{c}_t\|_{L^\infty(\Omega_T)} \leq R.$$

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With this information, we return to the a priori estimates in Theorem 5.7. Carefully reading through the proof of Theorem 5.7 reveals in detail how the constants of the a priori bounds are defined. More precisely, this shows that

\[ \|c\|_X \text{ is bounded in terms of \( \begin{cases} & \text{the data, the radius } R \text{ and the end time } T_0 \text{ if } c \neq \bar{c}, \\ & \text{the data and the end time } T_0 \text{ if } c = \bar{c}. \end{cases} \) } \]

In both cases, the constants of the a priori estimate are partially independent of the end time \( T_0 \). This means, we can split the constants into

\[ C = C(T_0) + C(T_0, R) + C_d. \]

We now choose the radius \( R := 2C(T_0) + 2C_d \). In the remaining part \( C(T_0, R) \), we assume a sufficiently small end time \( T_0 << 1 \) such that we have \( C(T_0, R) \leq C(T_0) + C_d \). This proves

\[ \|c\|_X \leq C = C(T_0) + C(T_0, R) + C_d \leq 2C(T_0) + 2C_d = R. \]

Thus, we have \( \mathcal{F}(K) \subset K \). However, we note that due to the assumption \( T_0 << 1 \), we are restricted to generally small time intervals \( I = [0, T_0] \).

Step 5 – weak*-continuity of \( \mathcal{F} \)

Subsequently, we use the already mentioned equivalence between weak*-continuous and sequentially weak*-continuous. This means, we show the weak*-continuity with the criterion based on sequences. For that purpose, we consider a sequence \( (\bar{c})_k \subset K \), for which we assume that \( \bar{c}_k \rightharpoonup \bar{c} \) in \( X \). As \( \mathcal{F}(\bar{c}_k) = c_k \) is the solution of (4.1e), we know together with \( \bar{c}_k \in K \) and the just established self-mapping property that

\[ \bar{c}_k \rightharpoonup \bar{c} \text{ in } X \text{ with } \bar{c} \in K \quad \text{and} \quad \|c_k\|_X = \|\mathcal{F}(\bar{c}_k)\|_X \leq R. \quad (5.15) \]

Consequently, \( (c_k)_k \) is a uniformly bounded sequence and a subsequence, denoted again by \( (c_k)_k \), weak*-converges to a unique limit \( c \in K \), i.e., it holds

\[ c_k \rightharpoonup c \text{ in } X \text{ with } c \in K. \quad (5.16) \]

The fixed point operator \( \mathcal{F} \) is weak*-continuous, if and only if \( c \) solves the “limit” PDE (4.1e), which is generated by \( \bar{c} \). In this case, we have

\[ \bar{c}_k \rightharpoonup \bar{c} \implies c_k = \mathcal{F}(\bar{c}_k) \rightharpoonup \mathcal{F}(\bar{c}) = c \text{ in } X. \quad (5.17) \]

Thus, it remains to show that \( c \) solves the “limit” PDE (4.1e), which is generated by \( \bar{c} \). To show this, we return to Definition 4.1 and we subtract the equations, which are generated by \( c_k \) from the equations, which are generated by \( \bar{c} \) and we integrate in time. Thereby, we obtain the error equations

**Gauss’s law:**

\[ (\mathcal{E}^{-1}(E_k - E), v)_{\Omega_T} = (\Phi_k - \Phi, \nabla \cdot v)_{\Omega_T}, \quad (5.18a) \]

\[ (\nabla \cdot (E_k - E), \varphi)_{\Omega_T} = \theta (\hat{\rho}_{f,k} - \hat{\rho}_f, \varphi)_{\Omega_T}. \quad (5.18b) \]

**Darcy’s law:**

\[ (K^{-1}(u_k - u), v)_{\Omega_T} = (\mu^{-1}(p_k - p), \nabla \cdot v)_{\Omega_T} + \theta \mu^{-1} (\hat{\rho}_{f,k} \mathcal{E}^{-1} E_k, v)_{\Omega_T} - \theta \mu^{-1} (\hat{\rho}_f \mathcal{E}^{-1} E, v)_{\Omega_T}, \quad (5.18c) \]

\[ (\nabla \cdot (u_k - u), \varphi)_{\Omega_T} = 0. \quad (5.18d) \]
Nernst–Planck equations:

\[
(\partial_t (c_l - c_l) , \varphi)_{L^2(I; H^1(\Omega))^*} + (D \nabla (c_l - c_l) , \nabla \varphi)_\Omega = 0
\]

for \(c_l,k \rightarrow c_l\) in \(L^2(\Omega_T)\) and \(c_l,k \rightarrow \bar{c}_l\) in \(L^2(I; L^2(\Omega))\).

We note that (5.15) and Aubin-Lions Lemma, cf. [39, Lemma 7.7], imply the norm-convergences

\[
\bar{c}_l,k \rightarrow \bar{c}_l \text{ in } L^2(\Omega_T) \quad \text{and} \quad \bar{c}_l,k \rightarrow \bar{c}_l \text{ in } L^2(I; L^2(\Omega)).
\]

Hence, we obtain for \((E, \Phi)\) analogously to the proof of Theorem 3.1 or Theorem 5.7, the norm-convergence \(\|\Phi_k - \Phi\|_{L^2(\Omega_T)} + \|E_k - E\|_{L^2(I; H^1(\Omega))} \leq C \sum \|\bar{c}_l,k - \bar{c}_l\|_{L^2(\Omega_T)} \rightarrow 0\).

Furthermore, for \((u, p)\), we get analogously to the proof of Theorem 3.1 or Theorem 5.7

\[
\|p_k - p\|_{L^2(\Omega_T)} + \|u_k - u\|_{L^2(I; H^1(\Omega))} \leq C \|\bar{p}_f,k E^{-1} E_k - \bar{p}_f E^{-1} E\|_{L^2(\Omega_T)}.
\]

Applying (A3), Hölder’s inequality, Lemma 4.4, and (5.19) yields for the right hand side

\[
C \|\bar{p}_f,k E^{-1} E_k - \bar{p}_f E^{-1} E\|_{L^2(\Omega_T)} \\
\leq C \|\bar{p}_f,k E^{-1} (E_k - E)\|_{L^2(\Omega_T)} + C \|\bar{p}_f,k - \bar{p}_f\|_{L^2(I; H^1(\Omega))} \|E\|_{L^2(I; H^1(\Omega))} \rightarrow 0.
\]

This shows the norm-convergence \(\|p_k - p\|_{L^2(\Omega_T)} + \|u_k - u\|_{L^2(I; H^1(\Omega))} \rightarrow 0\).

As to the convergence for \(c_k\), we begin with the time integrals and the diffusion integrals. From (5.16) follows that \(\partial_t \bar{c}_l,k \text{ resp. } \nabla \bar{c}_l,k \text{ weak}^*\text{-converge to } \partial_t \bar{c}_l \text{ in } L^2(I; H^1(\Omega))\) resp. \(\nabla \bar{c}_l \text{ in } L^2(\Omega_T)\). Thus, we have

\[
(\partial_t (c_l,k - c_l), \varphi)_{L^2(I; H^1(\Omega))}\times L^\infty(I; H^1(\Omega)) + (D \nabla (c_l,k - c_l), \nabla \varphi)_\Omega \rightarrow 0.
\]

For the convection integrals and the electric drift integrals, we obtain

\[
(c_l,k [u_k + e z_l (e k_b T)^{-1} E_k] - c_l [u + e z_l (e k_b T)^{-1} E], \nabla \varphi)_\Omega = 0
\]

Concerning 1.1, we note that \([u + e z_l (e k_b T)^{-1} E] \nabla \varphi \in L^1(\Omega_T)\) and \(c_l,k \rightarrow c_l \in L^\infty(\Omega_T)\) with \(L^\infty(\Omega_T) = L^1(\Omega_T)^*\). Thereby, we arrive with (5.16) at

\[
1.1 = \langle c_l,k - c_l, [u + e z_l (e k_b T)^{-1} E] \nabla \varphi \rangle_{L^1(\Omega_T)} \times L^1(\Omega_T) \rightarrow 0.
\]

Concerning 1.2, we come with Hölder’s inequality and the \(L^2(\Omega_T)\)-convergence of \(E_k\), \(u_k\) to

\[
1.2 \leq R \|\nabla \varphi\|_{L^2(\Omega_T)} \left(\|u_k - u\|_{L^2(\Omega_T)} + e z_l (e k_b T)^{-1} \|E_k - E\|_{L^2(\Omega_T)}\right) \rightarrow 0.
\]

Thus, we get for the convection integrals and the electric drift integrals

\[
(c_l,k [u_k + e z_l (e k_b T)^{-1} E_k] - c_l [u + e z_l (e k_b T)^{-1} E], \nabla \varphi)_\Omega \rightarrow 0.
\]

Finally, for \(c_k\) follows from (5.16) and Aubin-Lions Lemma the norm-convergence \(c_l,k \rightarrow c_l \text{ in } L^2(\Omega_T)\).

This leads for the reaction integrals with Hölder’s inequality immediately to

\[
\theta (R_l(c_k) - R_l(c)), \varphi)_\Omega \leq \theta \max_i C_{R_i} \|\varphi\|_{L^2(\Omega_T)} \sum_i \|c_l,k - c_l\|_{L^2(\Omega_T)} \rightarrow 0.
\]

\(^5\) In both cases weak”-convergence is equivalent to weak-convergence as the involved spaces are reflexive.
In summary, we have shown that \( c = \mathcal{F}(\bar{c}) \) for an arbitrarily chosen subsequence \((c_k)_k\). Therefore, the whole sequence \((c)_k\) converges and the operator \( \mathcal{F} \) is weak*-continuous in the sense of equation (5.17).

**Step 6 – existence:** A combination of Steps 1 – 5 shows that we can apply Theorem 5.10. This yields directly the existence of a solution \((E, \Phi, u, p, c)\) on a generally small time interval \([0, T_0]\).

We now consider for an arbitrary large end time \( \hat{T} \) a time interval \([0, \hat{T}]\), which we decompose with \((K + 1)\) time points \( 0 =: T_0 < T_1 < \ldots < T_K := \hat{T} \) into \( K \) subintervals \([T_i, T_{i+1}]\), \( i \in \{0, \ldots, K - 1\} \). Furthermore, we suppose that the subintervals \([T_i, T_{i+1}]\) are sufficiently small, such that Steps 1 – 5 are fulfilled. Thus, a local solution \((E_i, \Phi_i, u_i, p_i, c_i)\) exists on \([T_i, T_{i+1}]\) and this solution satisfies the a priori estimates from Theorem 5.7. We now carefully check how the constants of the a priori estimates depend on the end time \( T_{i+1} \) of the subinterval \([T_i, T_{i+1}]\). This reveals that the dependency of these constants on \( T_{i+1} \) behave as \( \exp(T_{i+1}) \), which eliminates any possibility of a blow-up on \([T_i, T_{i+1}]\). Thus, it is admissible to take the partial solution \( c_i \) as the initial value for the \((i + 1)\)-th solution \((E_{i+1}, \Phi_{i+1}, u_{i+1}, p_{i+1}, c_{i+1})\). This leads together with Theorem 3.1 to a continuation of the solution on the arbitrary large time interval \([0, \hat{T}]\) and consequently to a global solution.

However, we note that this continuation procedure does not lead to solutions on \([0, \infty)\). □

6 Conclusion

The contribution of this paper was to show the global existence of unique solutions of two-component electrolyte solutions, which are captured by the Darcy–Poisson–Nernst–Planck system. Here, two-component electrolyte solutions means that we considered electrolyte solutions, that consist of a neutral solvent and two oppositely charged solutes. In contrast to previous results, we allowed for two oppositely charged solutes with arbitrary valencies \( z_1 > 0 > z_2 \). Most importantly, we successfully established uniform a priori estimates for the chemical species by using weighted test functions, i.e., instead of the standard test test functions \( \varphi = \psi_l \), we used the weighted test functions \( \varphi = |z_l| \psi_l \). By means of this technique we avoided further restrictions such as the electroneutrality constraint of the volume-additivity constraint. Therefore, the results of this paper apply to general two-component electrolyte solutions, which are captured by the Darcy–Poisson–Nernst–Planck system. We note, that the a priori estimates include a uniform \( L^\infty(\Omega_T) \)-bound for the charged solutes \( c_l \), which we obtained by the use of Moser’s iteration technique. Moreover, the global existence and uniqueness result holds true in two space dimensions and three space dimensions.

To our best knowledge, in particular for the case of three spatial dimensions, this is the first global existence and uniqueness result for two-component electrolyte solutions, that firstly are governed by the Darcy–Poisson–Nernst–Planck system, that secondly include two oppositely charged chemical species with arbitrary valencies, and which thirdly are not subject to further restrictions such as the electroneutrality constraint, or the volume additivity constraint.

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**References**


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