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An existence result for a system of coupled semilinear diffusion-reaction equations with flux boundary conditions

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Abstract. In this paper, we consider diffusion, reaction and dissolution of mobile and immobile chemical species present in a porous medium. Inflow-outflow boundary conditions are considered at the outer boundary and the reactions amongst the species are assumed to be reversible which yield highly nonlinear reaction rate terms. The dissolution of immobile species takes place on the surfaces of the solid parts. Modeling of these processes leads to a system of coupled semilinear partial differential equations together with a system of ordinary differential equations with multivalued right hand sides. We prove the global existence of a unique positive weak solution of this model using some regularization technique, Schaefer’s fixed point theorem and Lyapunov type arguments.

1 Introduction

Crystal dissolution and precipitation have been extensively studied in the past years, e.g. see [2], [4], [5], [6], [15], [16], [18], [17] and references therein. In this work, we address the problem related to crystal (immobile species) dissolution and diffusion-reaction of mobile species which are present inside a porous medium. We consider the pore-scale model and prove the global existence of a unique positive weak solution. The mobile species (reactants) move via diffusion and advection inside the pore space and the bulk of the nonlinearities arise from the reactions among these species which are supposed to be reversible. The reactants interact with each other via equilibrium reactions and there is a supply of some or even all of them by dissolution from the surfaces of the solid parts. Forward as well as backward reactions are modelled by mass-action laws with arbitrarily large stoichiometric coefficients. The diffusion coefficients are supposed to be constant.

The dissolution of immobile species takes place on the surface of the solid parts inside the medium and the modeling leads to a system of ordinary differential equations (ODEs) with multivalued right hand side which adds a (small) difficulty and turns the whole problem into a weakly coupled pde-ode system. The coupling terms limit somehow the choice of spaces in which the concentrations should be looked for. It will turn out that the nonlinearities can be dealt with once one has some $L^\infty$-estimates for (prospective) solutions. This suggests to choose a solution space the components of which are imbedded into $L^\infty(S \times \Omega)$. The homogenization of this problem would be considered in our forthcoming paper.

1.1 The Model

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain which denotes the pore space inside a porous medium and its boundary is composed of two parts: the outer boundary $\partial \Omega$ and $\Gamma$ which is the
union of boundaries of the solid parts inside the given porous medium (cf. figure 1.1). We further assume that \( \partial \Omega := \Gamma_{\text{in}} \cup \Gamma_{\text{out}} \), where on \( \Gamma_{\text{in}} \) and \( \Gamma_{\text{out}} \) we prescribe the inflow and outflow boundary conditions respectively. \( S = (0, T) \) denotes the time interval with given \( T < \infty \). \( \Omega \) is assumed to be filled by some fluid with a-priori known Eulerian velocity \( \vec{q} = \vec{q}(t, x) \), \((t, x) \in S \times \Omega \) which satisfies
\[
\begin{align*}
\nabla \cdot \vec{q} &= 0 \quad \text{in } \Omega, \\
\vec{q} &= 0 \quad \text{on } \Gamma.
\end{align*}
\] (1.1)
with
\[
\begin{align*}
-\vec{q} \cdot \vec{n} > 0 \quad \text{on } \Gamma_{\text{in}}, \\
-\vec{q} \cdot \vec{n} \leq 0 \quad \text{on } \Gamma_{\text{out}}.
\end{align*}
\] (1.2)

![Figure 1.1: A porous medium containing mobile species in \( \Omega \) and immobile species on \( \Gamma \).](image)

Let \( I \) mobile species \( X_{\text{wd}}^i \) (type 1 species, \( i = 1, 2, \ldots, I \) and ” wd ” for without dissolution) be present in \( \Omega \). On \( \Gamma \) there are \( \bar{I} \) immobile species \( X_{\text{im}}^k \) (\( k = 1, 2, \ldots, \bar{I} \) and ” im ” for immobile) present. These immobile species dissolve and give \( I \) mobile species \( X_{\text{d}}^m \) (type 2 species, \( m = 1, 2, \ldots, \bar{I} \) and ” d ” for dissolution). Unifying the notation,
\[
X_i := \begin{cases} X_{\text{wd}}^i & \text{for } i = 1, 2, \ldots, I \\ X_{\text{d}}^i & \text{for } i = I + 1, I + 2, \ldots, I + \bar{I}. \end{cases} \] (1.3)

All the \( X_i \) diffuse, modelled by Fick’s law, and they undergo advection in \( \Omega \). Denote by \( w_l \) the concentration of \( X_{l}^l \), \( l = 1, \ldots, I \), by \( u_i \) the concentration of \( X_i \) and introduce the concentration vectors \( u = (u_1, \ldots, u_{I+\bar{I}}) \) and \( w = (w_1, \ldots, w_{\bar{I}}) \). Thus the total flux of species \( X_i \) inside \( \Omega \) is
\[
\vec{j}_i := -D_i \nabla u_i + \vec{q} u_i, \] (1.4)
where \( D_i \) is a constant. In this work we assume \( D := D_i > 0 \) for all \( i = 1, 2, \ldots, I + \bar{I} \).

**Remark 1.1.1.** The modeling of transport processes in a porous medium very often lead to the equations of type (1.7). In some situations the advective flux dominates diffusion and even though diffusion coefficients actually vary from species to species, we can consider the same value of the diffusion coefficients for all the species. Thus it makes sense to consider the same diffusion coefficients for all the species.

Let \( -\tau_{ij} \in \mathbb{Z}_0^- \) and \( \bar{\tau}_{ij} \in \mathbb{Z}_0^+ \) denote the corresponding stoichiometric coefficients of the corresponding \( J \) reversible reactions given by
\[
\sum_{i=1}^{I+\bar{I}} \tau_{ij} X_i = \frac{k^f_j}{k_b^j} \sum_{i=1}^{I+\bar{I}} \bar{\tau}_{ij} X_i, \quad j = 1, \ldots, J. \] (1.5)
Set $s_{ij} := \tau_{ij} - \tau_{ij}$ and $S = \{s_{ij}\}_{1 \leq i \leq I+\bar{I}}$. $k_f^j > 0$ and $k_b^j > 0$ are the forward and backward reaction rate factors respectively. The reaction rate term for $X_i$ is given by (via mass action law)

$$\left( SR(u) \right)_i = \sum_{j=1}^{J} s_{ij} R_j(u) = \sum_{j=1}^{J} s_{ij} \left( \prod_{r=1}^{I+\bar{I}} k_f^j \prod_{r=1}^{I+\bar{I}} u_r^{\sigma_{rj}} - k_b^j \prod_{r=1}^{I+\bar{I}} u_r^{\sigma_{rj}} \right), \quad (1.6)$$

The diffusion-reaction equations for the $i$-th mobile species is given by

$$\frac{\partial u_i}{\partial t} - \nabla \cdot (D \nabla u_i - \vec{q} u_i) = SR(u)_i \quad \text{in} \quad S \times \Omega. \quad (1.7)$$

for $i = 1, 2, ..., I + \bar{I}$. Let $d_k = d_k(t, x) \leq 0$ for $k = 1, 2, ..., I$. The boundary conditions are prescribed as

for $X_i^{\text{wd}}$: $j_k \cdot n = \begin{cases} d_k & \text{on} \ S \times \Gamma_{\text{in}}, \\ \vec{q} \cdot \vec{n} u_k & \text{on} \ S \times \Gamma_{\text{out}}, \\ 0 & \text{on} \ S \times \Gamma, \end{cases} \quad (1.8)$

$k = 1, 2, ..., I$ and

for $X_i^{d-l}$: $j_{I+l} \cdot n = \begin{cases} 0 & \text{on} \ S \times \Gamma_{\text{in}}, \\ \vec{q} \cdot \vec{n} u_{I+l} & \text{on} \ S \times \Gamma_{\text{out}}, \\ \frac{\partial w_l}{\partial t} & \text{on} \ S \times \Gamma, \end{cases} \quad (1.9)$

$l = 1, ..., \bar{I}$. The dissolution equation for the $l$-th immobile species is given as

$$\frac{\partial w_l}{\partial t} = -k_d z_l \quad \text{on} \ S \times \Gamma, \quad \text{(1.10)}$$

$$z_l \in \psi(w_l) \quad \text{on} \ S \times \Gamma, \quad \text{(1.11)}$$

where for simplicity $k_d (> 0)$ is assumed to be constant and

$$\psi(c) = \begin{cases} \{0\} & \text{if} \ c < 0, \\ [0, 1] & \text{if} \ c = 0, \\ \{1\} & \text{if} \ c > 0. \end{cases} \quad \text{(1.12)}$$

Following the same nondimensionalization technique as shown in [15], the complete system of diffusion-reaction equations is given as

$$\frac{\partial u_i}{\partial t} - \nabla \cdot (D \nabla u_i - \vec{q} u_i) = SR(u)_i \quad \text{in} \ S \times \Omega, \quad \text{(1.13)}$$

$$-(D \nabla u_k - \vec{q} u_k) \cdot \vec{n} = d_k \quad \text{on} \ S \times \Gamma_{\text{in}}, \quad \text{(1.14)}$$

$$-D \nabla u_k \cdot \vec{n} = 0 \quad \text{on} \ S \times \Gamma_{\text{out}}, \quad \text{(1.15)}$$

$$-D \nabla u_k \cdot \vec{n} = 0 \quad \text{on} \ S \times \Gamma, \quad \text{(1.16)}$$

$$-(D \nabla u_{I+l} - \vec{q} u_{I+l}) \cdot \vec{n} = 0 \quad \text{on} \ S \times \Gamma_{\text{in}}, \quad \text{(1.17)}$$

$$-D \nabla u_{I+l} \cdot \vec{n} = 0 \quad \text{on} \ S \times \Gamma_{\text{out}}, \quad \text{(1.18)}$$

\footnote{The proposed mathematical model is motivated from [15], [6], [8].}
\[ -D \nabla u_{I+1} \cdot \vec{n} = \varepsilon \frac{\partial w_l}{\partial t} \quad \text{on } S \times \Gamma, \quad (1.19) \]
\[ u_i(0, x) = u_0(x), \quad \text{in } \Omega, \quad (1.20) \]

where
\[ \frac{\partial w_l}{\partial t} = -k_d z_l \quad \text{on } S \times \Gamma, \quad (1.21) \]
\[ z_l \in \psi(w_l) \quad \text{on } S \times \Gamma, \quad (1.22) \]
\[ w_l(0, x) = w_0(x) \quad \text{on } \Gamma, \quad (1.23) \]

for \( i = 1, 2, \ldots, I + \bar{I}, k = 1, 2, \ldots, I \) and \( l = 1, 2, \ldots, \bar{I} \), where \( \varepsilon > 0 \) is a scale parameter and \( \psi \) is given by (1.12). For the velocity \( \vec{q} \), we have assumed:
\[ \nabla \cdot \vec{q} = 0 \text{ in } \Omega, \quad -\vec{q} \cdot \vec{n} > 0 \text{ on } \Gamma_{in}, \quad -\vec{q} \cdot \vec{n} \leq 0 \text{ on } \Gamma_{out} \text{ and } \vec{q} = 0 \text{ on } \Gamma. \quad (1.24) \]

In the next section we define some function spaces and collect some mathematical tools to investigate the above problem.

### 2 Mathematical Preliminaries

#### 2.1 Function Spaces

Let \( p > n + 2 \) be such that \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( \theta \in [0, 1] \). Assume that \( \Omega \subset \mathbb{R}^n \) (\( n \geq 2 \)) is a bounded domain with sufficiently smooth boundary \( \partial \Omega \) and \( \Gamma \). As usual \( L^p(\Omega), H^{1,p}(\Omega), C^\theta(\bar{\Omega}), (\cdot, \cdot)_{p,q} \) and \( [\cdot, \cdot]_\theta \) are the Lebesgue, Sobolev, Hölder, real- and complex-interpolation spaces respectively endowed with their standard norms.

For a Banach space \( X, X^* \) denotes its dual and the duality pairing is denoted by \( \langle \cdot, \cdot \rangle_{X^* \times X} \). We define the continuous embedding \( L^p(\Omega) \hookrightarrow H^{1,q}(\Omega)^* \) as
\[ \langle f, v \rangle_{H^{1,q}(\Omega)^* \times H^{1,q}(\Omega)} = \langle f, v \rangle_{L^p(\Omega) \times L^q(\Omega)} \text{ for } f \in L^p(\Omega), v \in H^{1,q}(\Omega). \quad (2.1) \]

The Sobolev-Bochner spaces are given by
\[ F := \left\{ u \in L^p(S; H^{1,p}(\Omega)) : \frac{du}{dt} \in L^p(S; H^{1,q}(\Omega)^*) \right\} \]
\[ := H^{1,p}(S; H^{1,q}(\Omega)^*) \cap L^p(S; H^{1,p}(\Omega)) \quad (2.2) \]

and
\[ M := \left\{ w \in L^p(S; L^p(\Gamma)) : \frac{dw}{dt} \in L^p(S; L^p(\Gamma)) \right\} = H^{1,p}(S; L^p(\Gamma)), \quad (2.3) \]

where \( \frac{d}{dt} \) is the distributional time derivative. These two spaces are furnished with the norms:
\[ ||u||_F := ||u||_{L^p(S; H^{1,p}(\Omega))} + ||u||_{L^p(S; H^{1,q}(\Omega)^*)} + \left| \left| \frac{du}{dt} \right| \right|_{L^p(S; H^{1,q}(\Omega)^*)} \quad (2.4) \]

\[^2\text{The symbols } \hookrightarrow, \hookrightarrow \hookrightarrow \text{ and } \overset{\cdot}{\rightarrow} \text{ denote the continuous, compact and dense embeddings, respectively.} \]
and
\[
||w||_M := ||w||_{L^p(S;L^p(\Gamma))} + \left|\left|\frac{dw}{dt}\right|\right|_{L^p(S;L^p(\Gamma))}.
\] (2.5)

Let \( I \in \mathbb{N} \). For \( a,b \in \mathbb{R}^I \), the scalar product and Euclidean norm are given by \( \langle a,b \rangle_I := \sum_{i=1}^{I} a_i b_i \) and \( ||a||_I := \sqrt{\sum_{i=1}^{I} |a_i|^2} \). Now we introduce the norms on the vector-valued function spaces. Assume \( u : \Omega \rightarrow \mathbb{R}^I \). We define
\[
[L^p(\Omega)]^I := \overbrace{L^p(\Omega) \times L^p(\Omega) \times \ldots \times L^p(\Omega)}^{I\text{-times}}
\]
and for \( u \in [L^p(\Omega)]^I \),
\[
||u||_{[L^p(\Omega)]^I} := \left[ \sum_{i=1}^{I} ||u_i||_{L^p(\Omega)}^{p} \right]^\frac{1}{p} \quad \text{for } p < \infty,
\]
\[
:= \max_{1 \leq i \leq I} ||u_i||_{L^\infty(\Omega)} \quad \text{for } p = \infty.
\] (2.6)

We also define
\[
F^{I+1} := \left[ H^{1,p}(S;H^{1,q}(\Omega)^*) \cap L^p(S;H^{1,p}(\Omega)) \right]^I,
\] (2.7)
\[
M^{I} := \left[ H^{1,p}(S;L^p(\Gamma)) \right]^I
\] (2.8)
and
\[
X_p^{I+I} := \left[ (H^{1,q}(\Omega)^*,H^{1,p}(\Omega))_{1-\frac{1}{p}} \right]^I.
\] (2.9)

The norms on \([H^{1,p}(\Omega)]^I\), \([H^{1,\infty}(\Omega)]^I\), \([H^{1,q}(\Omega)^*]^{I}\), \(F^{I+1}\), \(M^{I}\) and \(X_p^{I+I}\) are defined in the similar fashion as in (2.6).

**Theorem 2.1.1.** Let \( p > n+2 \), then \( F \hookrightarrow \leftrightarrow L^\infty(S \times \Omega) \).

**Proof.** Cf. theorem 2.2 in [10]. ♦

**Theorem 2.1.2.** Let \( p > n+2 \). Then \( (H^{1,q}(\Omega)^*,H^{1,p}(\Omega))_{1-\frac{1}{p}} \hookrightarrow \leftrightarrow L^\infty(\Omega) \).

**Proof.** Cf. theorem 2.3 in [10]. ♦

Let \( \{V,H,V^*\} \) be a Gelfand triple, where \( V \) a Banach space, \( H \) a Hilbert space and \( V^* \) is the dual of \( V \). Let \( H \) be identified with its own dual \( (H \cong H^*) \) and \( V \hookrightarrow \hookrightarrow H \), then \( H \hookrightarrow \hookrightarrow V^* \). Denote \( \Xi := \left\{ u \in L^p(S;V) : \frac{du}{dt} \in L^q(S;V^*) \right\} \). We have the following theorem:

**Theorem 2.1.3.** Let \( \{V,H,V^*\} \) be as above. Then \( \Xi \subset C([0,T];H) \) and the following rule of integration holds for any \( u,v \in \Xi \) and any \( 0 \leq t_1 \leq t_2 \leq T \):
\[
\int_{t_1}^{t_2} \frac{d}{dt}(u(t),v(t))_H \ dt = \int_{t_1}^{t_2} \langle \frac{du}{dt},v(t) \rangle_{V^* \times V} \ dt + \int_{t_1}^{t_2} \langle u(t),\frac{dv}{dt} \rangle_{V \times V^*} \ dt.
\] (2.10)

**Proof.** Cf. lemma 7.3 in [13]. ♦
**Theorem 2.1.4.** Let $B$ be a Banach space and $B_0$ and $B_1$ be reflexive spaces with $B_0 \subset B \subset B_1$. Suppose further that $B_0 \hookrightarrow B \hookrightarrow B_1$. For $1 < p,q < \infty$ and $0 < T < \infty$ define $X := \{u \in L^p(0,T;B_0) : \frac{\partial u}{\partial t} \in L^q(0,T;B_1)\}$. Then $X \hookrightarrow L^p(S;B)$.

**Proof.** See [14], pp. 106f, e.g.

Finally, $C$ and $C_i$ will denote nonnegative constants which may be different at different steps of the inequalities to come. Now to analyze the problem (1.13)-(1.24), our basic assumptions in this paper are:

(i) $p > n + 2$.  \hspace{1cm} (2.11)

(ii) $u_0$ and $w_0 \geq 0$ componentwise.  \hspace{1cm} (2.12)

(iii) $u_0, w_0 \in (H^1)^{j}((\Omega)\times(\Omega)), \frac{\partial w_0}{\partial t} \in L^\infty(\Gamma)$ for $i = 1,2,...,I + \bar{I}$ and $k = 1,2,...,\bar{I}$.  \hspace{1cm} (2.13)

(iv) All the reactions are linearly independent such that the stoichiometric matrix $S = (s_{ij})_{1 \leq i \leq I + \bar{I}, 1 \leq j \leq J}$ has the maximal column rank, i.e., $\text{rank}(S) = J$.  \hspace{1cm} (2.14)

$v$ $\bar{q}$ is the given fluid velocity which satisfies (1.24), $\bar{q} \in L^\infty(S \times \Omega)$. Define $Q := ||\bar{q}||_{L^\infty(S \times \Omega)}$. \hspace{1cm} (2.15)

(vi) $d_i \in L^\infty(S \times \Gamma_m)$ and $d_i \leq 0$ for all $i = 1,2,...,I$. \hspace{1cm} (2.16)

### 2.2 Regularization of $\psi(w)$

Multivaluedness in (1.22) motivates us to regularize $\psi$ by choosing a regularization parameter $\delta > 0$ as $^3$

$$
\psi_\delta(c) = \begin{cases} 
0 & \text{if } c \leq 0, \\
\frac{c}{\delta} & \text{if } 0 < c < \delta, \\
1 & \text{if } c \geq \delta.
\end{cases}
$$

We replace (2.11)-(2.13) by the following regularized ODE

$$
\frac{\partial w_\delta}{\partial t} = -k_d \psi_\delta(w_\delta) \quad \text{on } S \times \Gamma, \\
w_\delta(0,x) = w_0(x) \quad \text{on } \Gamma.
$$

**Lemma 2.2.1.** The problem (2.18)-(2.19) has a unique positive global weak solution $w_\delta \in H^{1,p}(S;L^p(\Gamma))$ which satisfies

$$
|||w_\delta|||_{L^p(S;L^p(\Gamma))} + \left|\left|\left| \frac{\partial w_\delta}{\partial t} \right|\right|_{L^p(S;L^p(\Gamma))} \right| \leq C,
$$

where $C$ is independent of $\delta$, $t$ and $w_\delta$.

**Proof.** The positivity of $w_\delta$ can be shown by testing (2.18) with $w_\delta^-$ (negative part of $w_\delta$).

Set $T_l(x) = \frac{w_0(x) - \delta}{k_d} - t$ for $l = 1,2,...,\bar{I}$. By direct computation the solution is given by

$$
w_\delta(t,x) = \begin{cases} 
w_0(x) - k_d t & \text{if } 0 \leq t \leq T_l(x) \\
e^{-\frac{w_0(x)}{\delta}} e^{\frac{w_0(x)}{\delta} k_d t} & \text{if } T_l(x) \leq t \leq T,
\end{cases}
$$

$^3$The function $\psi_\delta(w_\delta)$ is Lipschitz and monotonically increasing on $[0,\delta]$. 

for all \( l = 1, 2, \ldots, I \). Clearly \( w_{\delta_l} \) is measurable w.r.t. \( t \) and \( x \) and is in \( H^{1,p}(S;L^p(\Gamma))^I \).

Let \( w_{\delta_1}^1 \) and \( w_{\delta_2}^2 \) be the two solutions of (2.18)-(2.19). Set \( \bar{w}_{\delta} := w_{\delta_1}^1 - w_{\delta_2}^2 \). We test the ODE satisfied by \( w_{\delta} \) with \( \bar{w}_{\delta} \) and investing the knowledge of Lipschitz continuity of \( \psi_{\delta} \) and \( \bar{w}_{\delta}(0) = w_{\delta_1}^1(0) - w_{\delta_2}^2(0) = 0 \), then a straightforward application of Gronwall’s inequality yields the uniqueness of solution. The inequality (2.20) can be obtained by testing the \( l \)-th ODE of (2.18) by \( \frac{\partial w_{\delta_l}}{\partial t} |\frac{\partial w_{\delta_l}}{\partial t}|^{p-2} \) and \( w_{\delta_l} |w_{\delta_l}|^{p-2} \), respectively, where \( l = 1, 2, \ldots, I \).

We denote the problem (1.13)-(1.20) by \( P(u,SR(u)) \).

**Definition 2.2.2.** A function \( u \in F^{I+I} \) is said to be the weak solution of the problem (1.13)-(1.20) if it satisfies \( u(0,x) = u_0(x) \) and

\[
\langle \frac{\partial u(t)}{\partial t}, \phi \rangle_{H^{1,q}(\Omega)^I} + I + \sum_{i=1}^{I+I} \int_{\Omega} (D \nabla u_i(t,x) - \bar{q} u_i(t,x)) \cdot \nabla \phi_i(x) \, dx + \sum_{i=1}^{I+I} \int_{\Gamma_{in}} d_i(t,x) \phi_i(x) \, ds + \sum_{i=1}^{I+I} \int_{\Gamma_{out}} \bar{q} \cdot \bar{n} u_i(t,x) \phi_i(x) \, ds \geq \sum_{i=1}^{I+I} \int_{\Gamma_{out}} \bar{q} \cdot \bar{n} u_i(t,x) \phi_i(x) \, ds + \sum_{i=1}^{I+I} \int_{\Gamma} \frac{\partial u_i(t,x)}{\partial \nu} \phi_{I+i}(x) \, ds = \langle SR(u(t)), \phi \rangle_{H^{1,q}(\Omega)^I} \quad (2.22)
\]

for a.e. \( t \) and for every \( \phi \in H^{1,q}(\Omega)^I \).

Now we are ready to state the main theorem of this work which reads as

**Theorem 2.2.3.** Suppose that the assumptions (2.11)-(2.16) are satisfied, then there exists a unique positive global weak solution \( u \in F^{I+I} \) of the problem \( P(u,SR(u)) \).

Before dealing with the problem \( P(u,SR(u)) \), we consider a slightly modified problem and introduce the rate function \( \bar{R} : \mathbb{R}^{I+I} \rightarrow \mathbb{R}^{I+I} \) as

\[
\bar{R}(u) := R(u^+), \quad (2.23)
\]

where \( u^+ \) is the positive part of \( u \). We replace the reaction rate term \( SR(u) \) in \( P(u,SR(u)) \) by \( \bar{R}(u) \) and denote this modified problem by \( P(u,\bar{R}(u)) \). At first we will prove the existence of a global solution of \( P(u,\bar{R}(u)) \) and since it will be shown that the solution of \( P(u,\bar{R}(u)) \) is non-negative, it solves \( P(u,SR(u)) \). We commence our investigation with the solution of \( P(u,\bar{R}(u)) \).

**Lemma 2.2.4.** Let the assumptions (2.11)-(2.16) be satisfied and \( u \in F^{I+I} \) be the solution of \( P(u,\bar{R}(u)) \). Then \( u \geq 0 \) componentwise in \( S \times \Omega \).

**Proof.** Since for a.e. \( t, u(t) \in H^{1,q}(\Omega)^I \), we have \( u^-(t) \in H^{1,q}(\Omega)^I \). Testing the PDE (1.13) by \( u^-(t) \), we obtain

\[
\frac{1}{2} \frac{d}{dt} ||u^-(t)||^2_{L^2(\Omega)^I} + \frac{1}{2} \frac{d}{dt} ||\nabla u^-(t)||^2_{L^2(\Omega)^I} - \sum_{i=1}^{I+I} \int_{\Omega} \bar{q} \cdot \bar{n} u_i^-(t) u_i^-(t) \, dx - \sum_{i=1}^{I+I} \int_{\Gamma_{in}} d_i u_i^-(t) \, ds + \sum_{i=1}^{I+I} \int_{\Gamma_{out}} \bar{q} \cdot \bar{n} u_i^-(t) \, ds + \sum_{i=1}^{I+I} \int_{\Gamma_{out}} \bar{q} \cdot \bar{n} u_i^-(t) \, ds + \sum_{i=1}^{I+I} \int_{\Gamma} \varepsilon k_d z_i u_i^-(t) \, ds = - \int_{\Omega} \bar{R}(u(t)) u_i(t) \, dx,
\]
Lemma 3.1.1. Suppose that the assumptions (3.1) are satisfied, then there exists a unique positive global weak solution $u_{i} \in F^{I+I}$ of the problem $P(u_{i}, S\bar{R}(u_{i}))$. 

3 Existence of Solution of $P(u_{i}, S\bar{R}(u_{i}))$

Theorem 3.1. Suppose that the assumptions (2.11)-(2.16) are satisfied, then there exists a unique positive global weak solution $u_{i} \in F^{I+I}$ of the problem $P(u_{i}, S\bar{R}(u_{i}))$. 

3.1 An elementary lemma

Lemma 3.1.1. Suppose that $p > n + 2$, $x \in L^{\infty}(\Gamma_{m})$ and $d \in L^{\infty}(\Gamma_{m})$. If we define the maps $Q_{1}^{1} : [H^{1,p}(\Omega)]^{I} \rightarrow [H^{1,q}(\Omega)^{*}]^{I}$, $Q_{2}^{2} : [L^{\infty}(\Omega)]^{I} \rightarrow [H^{1,q}(\Omega)^{*}]^{I}$ and $R_{\Gamma} : [L^{p}(\Gamma)]^{I} \rightarrow [H^{1,q}(\Omega)^{*}]^{I}$ by

\[(i) \quad \langle Q_{1}^{1}(\phi), \xi \rangle := \sum_{k=1}^{I} \langle Q_{1}^{1}(\phi_{k}), \xi_{k} \rangle := \sum_{k=1}^{I} \int_{\Gamma_{m}} x \phi_{k} \xi_{k} ds \quad \text{for} \; \xi \in [H^{1,q}(\Omega)]^{I}, \]

\[(ii) \quad \langle Q_{2}^{2}(d), \xi \rangle := \sum_{i=1}^{I+I} \int_{\Gamma_{m}} d_{i} \xi_{i} ds \quad \text{for} \; \xi \in [H^{1,q}(\Omega)]^{I} \]

and

\[(iii) \quad \langle R_{\Gamma}(v), \eta \rangle := \sum_{k=1}^{I} \langle R_{\Gamma}(v_{k}), \eta_{k} \rangle := \sum_{k=1}^{I} \int_{\Gamma} v_{k}(x)\eta_{k}(x) ds \quad \text{for} \; \eta \in [H^{1,q}(\Omega)]^{I}. \]

Then $Q_{1}^{1}, Q_{2}^{2}$ and $R_{\Gamma}$ are continuous.

Proof. The proof follows by Hölder’s inequality and trace theorem. For instance, see lemma 5.15 in [12].
3.2 Introduction of a Lyapunov Functions

Let $r \in \mathbb{N}$ and $\mu^0 \in \mathbb{R}^{(I+\bar{I})}$ be a solution of the linear system

$$S^T \mu^0 = -\log K,$$

(3.4)

where $K \in \mathbb{R}^J$ is the vector of equilibrium constants $K_j = \frac{k_f^j}{k_b^j}$ related to the $J$ kinetic reactions. Due to assumption (2.14), the system (3.4) has a solution $\mu^0$. As in [7], we define the following functions:

Let $g_i : \mathbb{R}^+ \rightarrow \mathbb{R}$, $g : \mathbb{R}^0^{(I+\bar{I})} \rightarrow \mathbb{R}$, $f_r : \mathbb{R}^0^{(I+\bar{I})} \rightarrow \mathbb{R}$ and $F_r : L^\infty(\Omega)^{I+\bar{I}} \rightarrow \mathbb{R}$ be defined as

$$g_i(u_{i\delta}) = (\mu^0_i - 1 + \log u_{i\delta})u_{i\delta} + e^{(1-\mu^0_i)} \text{ for each } i = 1, 2, ..., I + \bar{I},$$

(3.5)

$$g(u_{\delta}) = \sum_{i=1}^{I+\bar{I}} g_i(u_{i\delta}),$$

(3.6)

$$f_r(u_{\delta}) = [g(u_{\delta})]^r$$

(3.7)

and

$$F_r(u_{\delta}) = \int_{\Omega} f_r(u_{\delta}(x)) \, dx.$$  

(3.8)

Proposition 3.2.1. Let $\alpha > 0$. There exist constants $C$ depending on $\alpha$ and $\mu_i$ but independent of $u_{\delta}$ such that

(i) $g(u_{\delta}) \geq g_i(u_{i\delta}) \geq u_{i\delta}$ and $F_r(u_{\delta}) \geq ||u_{\delta}||_{L^r(\Omega)},$

(3.9)

(ii) $g_i(u_{i\delta}) \leq C(1 + u_{i\delta}^{1+\alpha})$, $g(u_{\delta}) \leq C(1 + |u_{\delta}|^{1+\alpha})$ and $f_r(u_{\delta}) \leq C(1 + |u_{\delta}|^r(1+\alpha))$,  

(3.10)

for $i = 1, 2, ..., I + \bar{I}$.

Proof. The proof follows from the definitions of $g_i$, $g$, $f_r$ and $F_r$.

From (3.9) it is clear that the $L^r$-norm of $u_{\delta}$ will be finite if we can obtain an upper bound of $F_r(u_{\delta})$. For technical reason, we add an extra term on both sides of the PDE in the problem $P(u_{\delta}, S\tilde{R}(u_{\delta}))$ whereas the initial and boundary conditions remain unchanged, i.e., for any $\kappa > 0$, we have

$$\frac{\partial u_{\delta}}{\partial t} - \nabla \cdot (D\nabla u_{\delta} - \tilde{q}u_{\delta}) + \kappa u_{\delta} = S\tilde{R}(u_{\delta}) + \kappa u_{\delta} \quad \text{in} \quad S \times \Omega.$$  

(3.11)

We denote this modified problem by adding an extra term by $P_M(u_{\delta}, S\tilde{R}(u_{\delta}))$. Since a solution of $P_M(u_{\delta}, S\tilde{R}(u_{\delta}))$ is also a solution of $P(u_{\delta}, S\tilde{R}(u_{\delta}))$, at first we prove the global existence of the weak solution of $P_M(u_{\delta}, S\tilde{R}(u_{\delta}))$.

3.3 Derivation of A-priori Estimate

The main result of this section reads as

\[ \text{[Here we have considered the natural logarithm, i.e. } \log_e u_i.] \]
Theorem 3.3.1. Let \( r \in \mathbb{N} (r \geq 2) \) and \( 0 \leq t < T \). Further assume that \( u_\delta \in F^{I+I} \) is a solution of \( P_M(u_\delta, SR(u_\delta)) \). Then the following inequality holds good:

\[
F_r(u_\delta(t)) \leq e^{Ct} F_r(u_\delta(0)) \quad \text{for a.e. } t, \tag{3.12}
\]

where \( C \) is independent of \( \delta \) and \( t \).

To prove the above theorem we require the following lemma:

Lemma 3.3.2. Let \( p > n+2 \) and \( r \in \mathbb{N} (r \geq 2) \). Assume that \( u_\delta \in F^{I+I} \) is a solution of \( P_M(u_\delta, SR(u_\delta)) \) and for \( \tau > 0 \)

\[
u_{\delta, \tau} := u_\delta + \tau. \tag{3.13}
\]

Then the following inequality holds:

\[
\int_0^t \left( \frac{\partial u_\delta}{\partial \theta}, \partial f_r(u_{\delta, \tau}) \right)_{[H^1, q(\Omega)]^t \times H^1, q(\Omega)^{t+I}} d\theta \leq h(t, \tau, u_{\delta, \tau}) + l(t, \tau, u_{\delta, \tau}) + C \int_0^t F_r(u_{\delta, \tau}) d\theta \tag{3.14}
\]

for a.e. \( t \), where \( h(t, \tau, u_{\delta, \tau}) \) and \( l(t, \tau, u_{\delta, \tau}) \) tend to zero as \( \tau \to 0 \) for a.e. \( t \), and \( C \) is independent of \( \delta \), \( \tau \) and \( t \).

Remark 3.3.3. In [8] (see also [10]), it is shown that \( F_r : L^\infty(\Omega)^{I+I} \rightarrow \mathbb{R} \) and \( \partial f_r(u_{\delta, \tau}) : F^{I+I} \rightarrow L^\infty(S \times \Omega)^{I+I} \) are continuous. It can also be shown that \( \nabla_x (\partial f_r(u_{\delta, \tau})) \in L^q(S \times \Omega)^{I+I} \) and \( \partial f_r(u_{\delta, \tau}) \in L^q(S; H^1, q(\Omega))^{I+I} \).

Proof of Lemma 3.3.2. For \( p > n+2 \), \( u_{\delta, \tau} \in L^\infty(S \times \Omega)^{I+I} \) (by theorem 2.1.1) and \( \partial f_r(u_{\delta, \tau}) \in L^q(S; H^1, q(\Omega))^{I+I} \) (by lemma 4.1.1.6 in [10]). Using \( \partial f_r(u_{\delta, \tau}) \) in the weak formualtion of PDE (3.11), we get

\[
\int_0^t \left( \frac{\partial u_\delta}{\partial \theta}, \partial f_r(u_{\delta, \tau}) \right)_{[H^1, q(\Omega)]^t \times [H^1, q(\Omega)]^{t+I}} d\theta := I_{\text{diff}}^{(t)} + I_{\text{adv}}^{(t)} + I_{\text{ex}}^{(t)} + I_{\text{bound}}^{(t)} + I_{\text{reac}}^{(t)} \tag{3.15}
\]

where

\[
I_{\text{diff}}^{(t)} := - \sum_{l=1}^n \int_0^t \int_\Omega \left( D \frac{\partial}{\partial x^l} u_\delta, \frac{\partial}{\partial x^l} (\partial f_r(u_{\delta, \tau})) \right)_{I+I} d\theta, \tag{3.16}
\]

\[
I_{\text{bound}}^{(t)} := - \sum_{l=1}^I \int_0^t \int_{\Gamma l} (\lambda I_{\delta} - \vec{q} \cdot \vec{n} u_\delta)(\partial f_r(u_{\delta, \tau}))_{I+I} d\theta + \sum_{l=1}^I \int_\Omega \vec{q} \cdot \vec{n} u_\delta_{\delta, \tau} \partial f_r(u_{\delta, \tau})_{I+I} d\theta - \lambda \varepsilon \sum_{l=1}^I \int_0^t \int_{\Gamma l} \frac{\partial u_\delta}{\partial \theta} \partial f_r(u_{\delta, \tau})_{I+I} d\theta, \tag{3.17}
\]

\[
I_{\text{adv}}^{(t)} := - \sum_{l=1}^I \int_0^t \int_\Omega \vec{q} \cdot \nabla u_\delta \partial f_r(u_{\delta, \tau})_{I+I} d\theta, \tag{3.18}
\]

\[
I_{\text{ex}}^{(t)} := \lambda \int_0^t \langle S R(u_\delta), \partial f_r(u_{\delta, \tau}) \rangle_{[H^1, q(\Omega)]^t \times [H^1, q(\Omega)]^{I+I}} d\theta, \tag{3.19}
\]

\[
I_{\text{ex}}^{(t)} := - \kappa (1 - \lambda) \sum_{l=1}^I \int_0^t \int_\Omega u_\delta \partial f_r(u_{\delta, \tau})_{I+I} d\theta. \tag{3.20}
\]

Now we estimate the r.h.s. of (3.15). The terms \( I_{\text{diff}}^{(t)}, I_{\text{bound}}^{(t)}, I_{\text{adv}}^{(t)} \) and \( I_{\text{ex}}^{(t)} \) can be estimated in the similar fashion as shown in lemma 4.1.1.7 in [8] (see also [10]). To begin
with

\[ I_{\text{reac}}^{(t)} \leq \lambda \tau C \sum_{i=1}^{I} \int_0^t \int_{\Omega} \left[ \mu_0 + T |\log \tau| + u_{\delta,\tau} \right] dx d\theta =: h(t, \tau, u_{\delta,\tau}) \quad \text{for a.e. } t, \]

where \( C \) is independent of \( \epsilon, \delta, \tau, u_{\delta,\tau} \) and all the other terms of \( h(t, \tau, u_{\delta,\tau}) \) are bounded and tend to zero as \( \tau \to 0 \) for a.e. \( t \), i.e.,

\[ I_{\text{reac}}^{(t)} \leq h(t, \tau, u_{\delta,\tau}) \to 0 \quad \text{as } \tau \to 0 \quad \text{for a.e. } t. \quad (3.21) \]

\[ I_{\text{advec}}^{(t)} \leq C \int_0^t \int_{\partial \Omega} f_r(u_{\delta,\tau}) ds d\theta \quad \text{for a.e. } t, \quad (3.22) \]

where \( f_r \geq 0 \) and \( C := ||q \cdot n||_{L^\infty(S \times \Gamma_{in})} \) which is independent of \( \epsilon, \delta, \tau \) and \( t \).

\[ I_{\text{diff}}^{(t)} = -D \sum_{i=1}^{I} \sum_{n=1}^{I} \int_0^t \int_{\Omega} (\epsilon - 1) f_{r-2} \left( \mu_0 + \log u_{\delta,\tau} \right) \left( \mu_0 + \log u_{\delta,\tau} \right) \frac{\partial u_{\delta,\tau}}{\partial \xi_i} \frac{\partial u_{\delta,\tau}}{\partial x_1} dx d\theta \]

\[ \leq -D \sum_{i=1}^{I} \sum_{n=1}^{I} \int_0^t \int_{\Omega} (\epsilon - 1) f_{r-2} \left( \sum_{i=1}^{I} \left( \mu_0 + \log u_{\delta,\tau} \right) \frac{\partial u_{\delta,\tau}}{\partial x_1} \right)^2 dx d\theta \quad \text{for a.e. } t. \quad (3.23) \]

\[ I_{\text{bound}}^{(t)} = -\sum_{i=1}^{I} \int_0^t \int_{\Gamma_{in}} (\lambda \delta_i - q \cdot \bar{n} u_{\delta_i}) \partial f_r(u_{\delta,\tau}) ds d\theta + \sum_{i=1}^{I} \int_0^t \int_{\Gamma_{in}} q \cdot \bar{n} u_{\delta_i} \partial f_r(u_{\delta,\tau}) I_{\partial \Gamma} ds d\theta \]

\[ \leq C \left[ \int_0^t \int_{\Gamma_{in}} f_r(u_{\delta,\tau}) ds d\theta + \int_0^t \int_{\Gamma} f_r(u_{\delta,\tau}) ds d\theta \right] \]

\[ \leq C \int_0^t \int_{\Omega \times \Gamma} f_r(u_{\delta,\tau}) ds d\theta \]

\[ = C \int_0^t \left[ \left( \nabla f_2^2(u_{\delta,\tau}(t)) \right)^2_{L^2(\Omega \times \Gamma)} + \lambda \right] \left( \nabla f_2^2(u_{\delta,\tau}(t)) \right)^2_{L^2(\Omega)} d\theta, \quad (3.24) \]

where \( \varsigma \) and \( \Lambda \) are constants in Young’s inequality which will be chosen later. Furthermore
following [10] (see also lemma 4.1.2.7 in [9])

\[
I_{Ex}^{(t)} = -\kappa(1 - \lambda) \sum_{i=1}^{I+\bar{I}} \int_0^t \int_\Omega u_\delta \partial f_r(u_\delta, \tau) \, dx \, d\theta \\
= \kappa(1 - \lambda) \sum_{i=1}^{I+\bar{I}} \int_0^t \int_\Omega r(\tau - u_{\delta, \tau}) f_{r-1}(u_\delta, \tau)(\mu_i^0 + \log u_{\delta, \tau}) \, dx \, d\theta, \quad u_{\delta, \tau} = u_\delta + \tau \\
\leq r\delta \kappa IC \int_0^t \int_\Omega f_r(u_\delta) \, dx \, d\tau + r\kappa(I + \bar{I})(e(e - 1))^{-1} \int_0^t \int_\Omega f_r(u_\delta) \, dx \, d\tau \quad \text{for a.e. } t.
\] (3.25)

As \( \tau \to 0 \), \( f_r(u_{\delta, \tau}) \) is bounded in \( L^1(S \times \Omega) \). Therefore for a.e. \( t \) the first term in the r.h.s. of the above inequality tends to zero as \( \tau \to 0 \). Denote the first term by \( l(t, \tau, u_{\delta, \tau}) \), then

\[
I_{Ex}^{(t)} \leq l(t, \tau, u_{\delta, \tau}) + (I + \bar{I})r\kappa(e(e - 1))^{-1} \int_0^t \int_\Omega f_r(u_{\delta, \tau}) \, dx \, d\theta \quad \text{for a.e. } t.
\] (3.26)

Combining (3.15), (3.21), (3.22), (3.23), (3.24) and (3.26), we obtain

\[
\int_0^t \langle \partial_\theta u_\delta, \partial f_r(u_{\delta, \tau}) \rangle_{[H^1,\gamma(\Omega)]_{I+\bar{I}}} \, d\theta \\
\leq -Dr(r - 1) \sum_{i=1}^n \int_0^t \int_\Omega f_{r-2} \left( \sum_{i=1}^{I+\bar{I}} \left( \mu_i^0 + \log u_{\delta, \tau} \right) \frac{\partial u_{\delta, \tau}}{\partial x_1} \right)^2 \, dx \, d\theta \\
+ C\frac{r^2}{4} \int_0^t \int_\Omega f_{r-2} \left( \sum_{i=1}^{I+\bar{I}} \left( \mu_i^0 + \log u_{\delta, \tau} \right) \frac{\partial u_{\delta, \tau}}{\partial x_1} \right)^2 \, dx \, d\theta + C \int_0^t \int_\Omega \Lambda \left| f_{2}(u_{\delta, \tau}(t)) \right|^2 \, dx \, d\theta \\
+ h(t, \tau, u_{\delta, \tau}) + l(t, \tau, u_{\delta, \tau}) + (I + \bar{I})r\kappa(e(e - 1))^{-1} \int_0^t \int_\Omega f_r(u_{\delta, \tau}) \, dx \, d\theta \\
\leq \left[ -Dr(r - 1) + C\frac{r^2}{4} \right] \int_0^t \int_\Omega f_{r-2} \left( \sum_{i=1}^{I+\bar{I}} \left( \mu_i^0 + \log u_{\delta, \tau} \right) \frac{\partial u_{\delta, \tau}}{\partial x_1} \right)^2 \, dx \, d\theta \\
+ h(t, \tau, u_{\delta, \tau}) + l(t, \tau, u_{\delta, \tau}) + (I + \bar{I})r\kappa(e(e - 1))^{-1} \int_0^t \int_\Omega f_r(u_{\delta, \tau}) \, dx \, d\theta
\] (3.27)

Choosing \( \varsigma \leq \frac{4D(r-1)}{Cr} \), this shows that \( \Lambda \) is independent of \( \lambda, \delta \) and \( \tau \). This gives

\[
\int_0^t \langle \partial_\theta u_\delta, \partial f_r(u_{\delta, \tau}) \rangle_{[H^1,\gamma(\Omega)]_{I+\bar{I}}} \, d\theta \\
\leq h(t, \tau, u_{\delta, \tau}) + l(t, \tau, u_{\delta, \tau}) + (C\Lambda + (I + \bar{I})r\kappa(e(e - 1))^{-1}) \int_0^t \int_\Omega f_r(u_{\delta, \tau}) \, dx \, d\theta \\
\leq h(t, \tau, u_{\delta, \tau}) + l(t, \tau, u_{\delta, \tau}) + C \int_0^t F_r(u_{\delta, \tau}) \, d\theta \quad \text{for a.e. } t.
\] (3.28)

where \( C \left( := (C\Lambda + (I + \bar{I})r\kappa(e(e - 1))^{-1}) \right) \), and \( h(t, \tau, u_{\delta, \tau}) \) and \( l(t, \tau, u_{\delta, \tau}) \) tend to zero as \( \tau \to 0 \) for a.e. \( t \). \hfill \diamond

**Proof of theorem 3.3.1.** Let \( u_\delta \) be a solution of the problem \( P_M(u_\delta, S\bar{R}(u_\delta)) \). Since we only the know the nonnegativity of \( u_\delta \), let \( u_{\delta, \tau} := u_\delta + \tau \) for \( \tau > 0 \). Clearly \( u_{\delta, \tau} \in F^{I+\bar{I}} \).
Replicating the steps of theorem 3 in [8] (see also theorem 4.1.1.3 in [10]), we obtain

\[ F_r(u_{\delta}(t)) \leq F_r(u_{\delta}(0)) + C \int_0^t F_r(u_{\delta}) \, d\theta \quad \text{for a.e. } t. \]

Here we have used the properties that \( u_{\delta,\tau} \to u_{\delta} \), \( h(t, \tau, u_{\delta,\tau}) \to 0 \) and \( l(t, \tau, u_{\delta,\tau}) \) as \( \tau \to 0 \) for a.e. \( t \) and \( F_r(u_{\delta,\tau}) \) is continuous (cf. remark 3.3.3). Gronwall’s inequality gives

\[ F_r(u_{\delta}(t)) \leq e^{Ct} F_r(u_{\delta}(0)) \quad \text{for a.e. } t. \]  

(3.29)

where \( C \) is independent of \( \delta, \tau \) and \( t \). This establishes the inequality (3.12).

Now we use theorem 3.3.1 to obtain \( L^r \)- and \( L^\infty \)-estimates of the solution \( u_{\delta} \).

\textbf{Corollary 3.3.4.} For any arbitrary solution \( u_{\delta} \in F^{l+\bar{I}} \) of \( P_M(u_{\delta}, SR(u_{\delta})) \) the following estimates hold true:

\[
\sup_{\delta > 0} \|\| u_{\delta}(t) \|\|_{L^r(\Omega)^{l+\bar{I}}} \leq C < \infty \quad \text{for all } r \text{ and for a.e. } t
\]

(3.30)

and

\[
\sup_{\delta > 0} \|\| u_{\delta}(t) \|\|_{L^\infty(\Omega)^{l+\bar{I}}} \leq C < \infty \quad \text{for a.e. } t.
\]

(3.31)

\textbf{Proof.} The proof follows like the one for lemma 6 in [8]. See also corollary 4.1.1.8 in [10]. ♦

\textbf{Corollary 3.3.5.} Let the assumptions (2.11)-(2.16) and \( r \in \mathbb{N} \) be satisfied. Then there exists a constant \( C \) independent of \( u_{\delta}, \delta \) and \( t \) such that any arbitrary solution \( u_{\delta} \in F^{l+\bar{I}} \) of the problem \( P_M(u_{\delta}, SR(u_{\delta})) \) satisfies

\[
\|\| u_{\delta} \|\|_{F^{l+\bar{I}}} \leq C.
\]

(3.32)

\textbf{Proof.} Note that \( u_{\delta} \) satisfies the estimates (3.30) and (3.31). The abstract formulation of the problem \( P_M(u_{\delta}, SR(u_{\delta})) \) is given by

\[
\frac{\partial u_{\delta}}{\partial t} + Au_{\delta} = f_{Bd}(u_{\delta}) + f_{Ex}(u) + f(u_{\delta}),
\]

(3.33)

\[
u_{\delta}(0, x) = u_0(x),
\]

(3.34)

where the operator \( A: [H^{1,p}(\Omega)]^{l+\bar{I}} \rightarrow [H^{1,q}(\Omega)]^{l+\bar{I}} \) is defined as \( Au := (A_1 u_1, A_2 u_2, ..., A_{l+\bar{I}} u_{l+\bar{I}}) \) such that for \( 1 \leq i \leq l+\bar{I} \)

\[
(A_i u_i, u_i) := \int_{\Omega} D^{i} u_i(x) \cdot \nabla u_i(x) \, dx + \kappa \int_{\Omega} u_i(x) \, dx \quad \text{for } u_i \in H^{1,p}(\Omega), \kappa_i \in H^{1,q}(\Omega).
\]

(3.35)

By theorem 5.6 in [12] (see also section 3 in [1]), \( A \) has maximal regularity on \([H^{1,q}(\Omega)]^{l+\bar{I}}\). The term \( f(u_{\delta}) = SR(u_{\delta}) + \kappa u_{\delta}, f_{Ex}(u_{\delta}) := -q^{\top} \nabla u_{\delta} \) and \( f_{Bd}(u_{\delta}) = Q_{in}^2 u_{\delta} + Q_{in}^2 (-d) + R_{in}(-\varepsilon \partial^2_{\nu u_{\delta}}). \) Choosing \( r \) sufficiently large in (3.30) and application of Hölder’s inequality imply that \( f \in L^p(\bar{S}; [H^{1,q}(\Omega)]^{l+\bar{I}}). \) Using lemma 3.1.1, we get \( f_{Bd} \in L^p(\bar{S}; [H^{1,q}(\Omega)]^{l+\bar{I}}) \) and by Hölder’s inequality \( f_{Ex} \in L^p(\bar{S}; [H^{1,q}(\Omega)]^{l+\bar{I}}). \) Moreover from (2.25), we have \( u_0 \in [(H^{1,q}(\Omega), H^{1,p}(\Omega))]_{1-\frac{1}{p},p}^{l+\bar{I}}. \) Therefore from theorem 2.5 in [11] there exists a unique
\( u_\delta \in F^{I + \bar{I}} \) such that
\[
|||u_\delta|||_{F^{I + \bar{I}}} \leq C \left(|||u_0|||_{X_{0}^{p} + \bar{I}} + |||f_{Bd} + f_{Ex} + f|||_{L^{p}(\mathcal{S};H^{1,q}(\Omega))^n} \right) \leq C,
\tag{3.36}
\]
where \( C \) is independent of \( u_\delta \) and \( \delta \).

### 3.4 Existence of solution of \( P(u_\delta, \bar{S}R(u_\delta)) \)

We employ Schaefer’s fixed point theorem to prove the existence. Let (2.11)-(2.16) hold. Define the fixed point operator \( Z : F^{I + \bar{I}} \to F^{I + \bar{I}} \) by \( u_\delta := Z(v) \), where \( u_\delta \) is the solution of
\[
\frac{\partial u_\delta}{\partial t} - \nabla \cdot (D \nabla u_\delta \cdot \bar{q}_u) + \kappa u_\delta = S \bar{R}(v) + \kappa v \quad \text{in} \; S \times \Omega,
\tag{3.37}
\]
\[
-D \bar{u}_k \cdot n = \delta_k \quad \text{on} \; S \times \Gamma_{in},
\tag{3.38}
\]
\[
-D \bar{u}_{\delta_l} \cdot n = 0 \quad \text{on} \; S \times \Gamma_{out},
\tag{3.39}
\]
\[
-D \nabla u_{\delta_{l+1}} \cdot n = 0 \quad \text{on} \; S \times \Gamma_{in},
\tag{3.40}
\]
\[
-D \nabla u_{\delta_{l+1}} \cdot n = 0 \quad \text{on} \; S \times \Gamma_{out},
\tag{3.41}
\]
\[
\bar{u}_{\delta_{0}}(0, x) = u_{0}(x), \quad \text{in} \; \Omega.
\tag{3.42}
\]
for \( i = 1, 2, ..., I + \bar{I} \), \( k = 1, 2, ..., I \) and \( l = 1, 2, ..., \bar{I} \). Note that \( u_\delta \) is the solution of (2.18)-(2.19). Every fixed point of \( Z \) is a solution of \( P_{M}(u_\delta, \bar{S}R(u_\delta)) \). Let us verify that \( Z \) is well defined. The abstract formulation of (3.37)-(3.44) is given by
\[
\frac{\partial u_\delta}{\partial t} + Au_\delta = f(v) + f_{Bd}(u_\delta) + f_{Ex}(u_\delta),
\tag{3.45}
\]
\[
u_{\delta_{0}}(0, x) = u_{0}(x),
\tag{3.46}
\]
where the operator \( A : H^{1,p}(\Omega)^{I + \bar{I}} \to [H^{1,q}(\Omega)]^{I + \bar{I}} \) is defined as in corollary 3.3.5 and has maximal regularity on \( [H^{1,q}(\Omega)]^{I + \bar{I}} \). The term \( f(v) = S \bar{R}(v) + \kappa v \), \( f_{Bd}(u_\delta) := -\bar{q} \cdot \nabla \bar{u}_\delta \) and \( f_{Ex}(u_\delta) = Q_{\in}(u_\delta) + Q_{\in}^{2}(d) + R_{\in}(\bar{q} \cdot \nabla \bar{u}_\delta) \). For \( p > n + 2 \) \( v \in L^{\infty}(\Omega)^{I + \bar{I}} \) and this gives \( f \in L^{p}(S;H^{1,q}(\Omega)^{I + \bar{I}}) \). Using lemma 3.1.1, we get \( f_{Bd} \in L^{p}(S;H^{1,q}(\Omega)^{I + \bar{I}}) \) and by Hölder’s inequality \( f_{Ex} \in L^{p}(S;H^{1,q}(\Omega)^{I + \bar{I}}) \). Since \( u_0 \in [H^{1,q}(\Omega)^{n},H^{1,q}(\Omega)]_{1-p}^{1-p} \bar{I} + \bar{I} \), therefore from theorem 2.5 in [11] there exists a unique \( u_\delta \in F^{I + \bar{I}} \). Thus \( Z \) is well defined. The same steps show that \( Z \) is continuous and compact.

**Lemma 3.4.1.** There exists a unique positive global weak solution of the \( P(u_\delta, \bar{S}R(u_\delta)) \).

**Proof.** We employ Schaefer’s fixed point theorem. From above it is clear that the operator \( Z \) is continuous and compact. We remain to check that the set \( \{ u_\delta \in F^{I + \bar{I}} : \exists \lambda \in [0,1] \text{ s.t. } u_\delta = \lambda Z(u_\delta) \} \) is bounded, i.e., we need to obtain an estimate of the solution of
\[
\frac{\partial u_\delta}{\partial t} - \nabla \cdot (D \nabla u_\delta - \bar{q}_u) + \kappa u_\delta = \lambda \bar{S}R(u_\delta) + \lambda \kappa u_\delta,
\tag{3.47}
\]
where initial and boundary values \( u_0, d \) and \( \bar{q} \cdot \nabla \bar{u}_\delta \) are replaced by \( \lambda u_0, \lambda d \) and \( \lambda \bar{q} \cdot \nabla \bar{u}_\delta \), respectively. Clearly the estimates in section 3.3, derived for \( \lambda = 1 \), hold for \( 0 \leq \lambda \leq 1 \). Thus \( Z \) has a fixed point, i.e., there exists a solution of \( P_{M}(u_\delta, \bar{S}R(u_\delta)) \). This implies
Now we have sufficient tools to send \( \delta \to 0 \). We follow the idea shown in [15]. Let \( z_\delta \in L^\infty(S \times \Gamma) \) be defined by
\[
z_\delta(t,x) = \psi_\delta(w_\delta(t,x)) \quad \text{for a.e.} \ (t,x) \in S \times \Gamma.
\] (4.2)
Due to estimates (2.20) and (4.1), up to a subsequence, following convergences hold:
\[
\begin{align*}
(i) \quad & u_\delta \rightharpoonup u & \text{in} & & L^2(S; H^{1,2}(\Omega))^{I+\bar{I}}. \\
(ii) \quad & \frac{\partial u_\delta}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} & \text{in} & & L^2(S; H^{1,2}(\Omega)^*)^{I+\bar{I}}. \\
(iii) \quad & w_\delta \rightharpoonup w & \text{in} & & L^p(S \times \Gamma)^{\bar{I}}. \\
(iv) \quad & \frac{\partial w_\delta}{\partial t} \rightharpoonup \frac{\partial w}{\partial t} & \text{in} & & L^p(S \times \Gamma)^{\bar{I}}. \\
(v) \quad & z_\delta \rightharpoonup z & \text{in} & & L^\infty(S \times \Gamma)^{\bar{I}}.
\end{align*}
\] (4.3)

**Lemma 4.1.2.** Up to a subsequence, \( (u_\delta)_{\delta>0} \) is strongly convergent to \( u \) in \( L^2(S; L^2(\Omega))^{I+\bar{I}} \).

**Proof.** The proof follows from the estimate (4.1) and theorem 2.1.4. ♦

**Lemma 4.1.3.** The weak limit \( u \) belongs to \( L^\infty(S \times \Omega)^{I+\bar{I}} \).

**Proof.** Since \( (u_\delta)_{\delta>0} \) is strongly convergent to \( u \) in \( L^2(S; L^2(\Omega))^{I+\bar{I}} \), there exists a subsequence \( (u_{\delta'})_{\delta'>0} \) which is pointwise convergent to \( u \) almost everywhere in \( S \times \Omega \) (see corollary on page 53 in [19]), i.e.,
\[
\lim_{\delta' \to 0} u_{\delta'}(t,x) = u(t,x) \quad \text{a.e.} \quad (t,x) \in S \times \Omega.
\] ♦

4 Proof of theorem 2.2.3

4.1 Passage to the Limit as \( \delta \to 0 \)

**Theorem 4.1.1.** For any \( \delta > 0 \), the solution \( u_\delta \) of the problem \( P(u_\delta, S\bar{R}(u_\delta)) \) satisfies the following estimate:
\[
||| u_\delta |||_{L^2(S; L^2(\Omega))^{I+\bar{I}}} + ||| \nabla u_\delta |||_{L^\infty(S; L^\infty(\Omega))^{I+\bar{I}}} + ||| \nabla u_\delta |||_{L^2(S; L^2(\Omega))^{I+\bar{I}}} + ||| \frac{\partial u_\delta}{\partial t} |||_{L^2(S; H^{1,2}(\Omega)^*)^{I+\bar{I}}}
\leq C < \infty,
\] (4.1)
where \( C \) is independent of \( \delta \) and \( u_\delta \).

**Proof.** The proof is obtained by the classical energy method. The estimates for \( u_\delta \) and \( \nabla u_\delta \) follow by testing the PDE in \( P(u_\delta, S\bar{R}(u_\delta)) \) with \( u_\delta \). The estimate for \( \frac{\partial u_\delta}{\partial t} \) is obtained by using a \( \phi \) in \( L^2(S; H^{1,2}(\Omega))^{I+\bar{I}} \) as the test function. The \( L^\infty \) bound for \( u_\delta \) follows from corollary 3.3.4. ♦
By (4.1) \( \|u_\delta\|_{L^\infty(S; L^\infty(\Omega))} \leq C < \infty \) for all \( i \), therefore

\[
|u_i(t, x)|^2 \leq \sum_{i=1}^{I+I} |u_i(t, x)|^2 = \lim_{\delta' \to 0} \sum_{i=1}^{I+I} \left| u_{\delta'}(t, x) \right|^2
\]

\[
\leq \sum_{i=1}^{I+I} \lim_{\delta' \to 0} \operatorname{ess sup}_S \operatorname{sup}_\Omega \left| u_{\delta'}(t, x) \right|^2
\]

\[
\leq \sum_{i=1}^{I+I} \lim_{\delta' \to 0} \operatorname{ess sup} C \text{ for a.e. } t \text{ and } x
\]

\[
\operatorname{ess sup}_{S \times \Omega} |u_i(t, x)| \leq C < \infty
\]

This gives

\[
\|u\|_{L^\infty(S \times \Omega)^{I+I}} = \max_{1 \leq i \leq I+I} \|u_i\|_{L^\infty(S \times \Omega)} = \max_{1 \leq i \leq I+I} \operatorname{ess sup}_{S \times \Omega} |u_i(t, x)| \leq C < \infty.
\]

**Lemma 4.1.4.** The source term \((S\bar{R}(u_\delta))_{\delta > 0}\) is strongly convergent to \(S\bar{R}(u)\) in \(L^2(S \times \Omega)^{I+I}\).

*Proof.* The strong convergence of \((u_\delta)_{\delta > 0}\) and \(L^\infty\)-estimates of \(u_\delta\) and \(u\) finish off the proof. ♦

**Lemma 4.1.5.** There exists \(w \in H^{1,p}(S; L^p(\Gamma))^I\) and \(z \in L^\infty(S \times \Gamma)^I\) which satisfies

\[
\frac{\partial w}{\partial t} = -kdz \quad \text{on } S \times \Gamma,
\]

\[
z \in \psi(w) \quad \text{on } S \times \Gamma,
\]

\[
w(0, x) = w_0(x) \quad \text{on } \Gamma,
\]

\[
(4.4) \quad (4.5) \quad (4.6) \quad (4.7)
\]

where

\[
\psi(c) = \begin{cases} 
0 & \text{if } c < 0, \\
[0, 1] & \text{if } c = 0, \\
\{1\} & \text{if } c > 0,
\end{cases}
\]

(4.8)

and the following estimate hold:

\[
\|w\|_{L^p(S \times \Gamma)^I} + \|w\|_{L^p(S \times \Gamma)^I} + \left\| \frac{\partial w}{\partial t} \right\|_{L^p(S \times \Gamma)^I} \leq C < \infty,
\]

(4.9)

where \(C\) is independent of \(\delta\) and \(w_\delta\).

*Proof.* The *a-priori* estimate (4.9) follows from the convergences in (4.3). Here special attention needs to be paid to prove (4.8). This part is shown in theorem 2.21 in [15]. ♦

**Theorem 4.1.6.** There exists a unique weak solution \(u \in [H^{1,2}(S; H^{1,2}(\Omega))^*) \cap L^2(S; H^{1,2}(\Omega))^* \cap L^2(S; H^{1,2}(\Omega))^* \cap L^2(S; H^{1,2}(\Omega))^* \cap L^2(S; H^{1,2}(\Omega))^* \cap \)
\[ L^\infty(S \times \Omega)^{I+I} \] of the problem \( P(u,SR(u)) \) which satisfies

\[
||u||_{L^2(S;L^2(\Omega))^{I+I}} + ||u||_{L^\infty(S;L^\infty(\Omega))^{I+I}} + ||\nabla u||_{L^2(S;L^2(\Omega))^{I+I}} + \left\| \frac{\partial u}{\partial t} \right\|_{L^{p}(S;H^{1,q}(\Omega)^{I+I})^{I+I}} \leq C < \infty,
\]

where \( C \) is independent of \( \delta \) and \( u_\delta \).

**Proof.** The estimate (4.10) follows immediately from the weak convergences in (4.3) and the rest is shown in theorem 2.21 in [15].

**Theorem 4.1.7.** There exists a unique positive global weak solution \( u \in [H^1,p(S;H^1,q(\Omega)^*) \cap L^p(S;H^1,p(\Omega)) \cap L^\infty(S \times \Omega)]^{I+I} \) for \( P(u,SR(u)) \).

**Proof.** The abstract formulation of the problem \( P(u,SR(u)) \) is given by

\[
\frac{\partial u}{\partial t} + Au = f(u) + f_{BD}(u) + f_{Ex}(u),
\]

\[
u(0,x) = u_0(x),
\]

where the operator \( A \) is defined as in remark 3.3.5 with maximal regularity on \([H^1,q(\Omega)^*]^{I+I}\), \( f(u) = SR(u) + \kappa u \), \( f_{Ex}(u) := -q \cdot \nabla u \) and \( f_{BD}(u) = Q_{1u}(u) + Q_{2u}(-d) + R_{f}(\varepsilon,\frac{\partial u}{\partial t}) \).

Using (4.10) and Hölder’s inequality imply that \( f \in L^p(S;H^1,q(\Omega)^*)^{I+I} \).

Using lemma 3.1.1, we get \( f_{BD} \in L^p(S;H^1,q(\Omega)^*)^{I+I} \) and by Hölder’s inequality \( f_{Ex} \in L^p(S;H^1,q(\Omega)^*)^{I+I} \). Moreover from (2.25), we have \( u_0 \in [(H^1,q(\Omega)^*;H^1,p(\Omega))]^{I+I} \).

Therefore from theorem 2.5 in [11] there exists a unique \( u \in F^I \) such that

\[
||u||_{F^{I+I}} \leq C \left( ||u_0||_{X^{I+I}_p} + ||f_{BD} + f_{Ex} + f||_{L^p(S;H^1,q(\Omega)^*)^{I+I}} \right) \leq C,
\]

where \( C \) is independent of \( u \) and \( p \). This completes the proof.

**Proof of theorem 2.2.3.** Theorem 4.1.7 implies that there exists a unique weak solution \( u \in F^{I+I} \) for \( P(u,SR(u)) \). Since the solution of \( P(u,SR(u)) \) is non-negative, it solves \( P(u,SR(u)) \), i.e., there exists a unique positive global weak solution \( u \in F^{I+I} \) for \( P(u,SR(u)) \).

\[ \dashv \]

## 5 Appendix

### 5.1 Some important theorems and inequalities

**Theorem 5.1.1** (Schaefer’s fixed point theorem). Let \( X \) be Banach space. Assume that \( Z : X \to X \) is a continuous and compact mapping and the solution of the set

\[
\{ u \in X | \exists \lambda \in [0,1] : u = \lambda Z(u) \}
\]

is bounded. Then \( Z \) has a fixed point.

**Proof.** See theorem 4 in section 9.2.2 in [3].

**Proof.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with sufficiently smooth boundary \( \partial \Omega \), then for all \( u \in H^{1,2}(\Omega) \) the following estimate hold:

\[
||u||_{L^2(\partial \Omega)}^2 \leq C ||u||_{H^{1,2}(\Omega)} ||u||_{L^2(\Omega)},
\]

(5.1)
where the constant $C$ is independent of $u$.

Proof. See lemma 5.6 in [7].

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