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ANALYSIS OF A COMBINED CG1-DG2 METHOD FOR THE TRANSPORT EQUATION

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Abstract.
In this paper, we introduce a reduced discontinuous Galerkin method in which the space of continuous piecewise-linear functions (CG1) is enriched with discontinuous piecewise-quadratics (DG2). The resultant finite element approximation is continuous at the vertices of the mesh and discontinuous across edges/faces. We analyze the properties of the CG1-DG2 discretization in the context of a steady linear transport equation. The presented a priori error estimate shows that the discontinuous enrichment stabilizes the continuous coarse-scale component and delivers optimal convergence rates. Numerical studies for steady and unsteady convection problems confirm this result.

Key words. convective transport, finite element discretization, reduced discontinuous Galerkin method, a priori error estimates

AMS subject classifications. 65N12, 65N15, 65N30, 65N50

1. Introduction. The design of finite element methods for convection-dominated transport problems requires a careful choice of approximation spaces and/or certain modifications of the variational formulation (see, e.g., [8, 17] and references therein). The continuous Galerkin (CG) method is computationally efficient but numerically unstable in the limit of small or vanishing diffusion [10]. This instability manifests itself in suboptimal convergence rates and in the tendency of CG to produce nonphysical oscillations (“wiggles”) in the neighborhood of unresolvable small-scale features. The standard CG discretization can be stabilized, e.g., using streamline diffusion, modified test functions, residual-free bubbles or interior penalty terms. In addition to linear stabilization, nonlinear shock-capturing dissipation is commonly employed when it comes to solving hyperbolic conservation laws and systems thereof.

In discontinuous Galerkin (DG) methods [2, 11, 7, 16], the use of upwind-biased convective fluxes leads to an approximation which is intrinsically stable and exhibits optimal convergence rates in applications to problems with smooth solutions. However, the treatment of diffusive terms becomes more involved, and the number of degrees of freedom is exorbitant compared to CG approximations. Hughes et al. [9] developed a multiscale DG method in which the computational cost is reduced using an additive decomposition of the finite element space into continuous coarse scales and discontinuous fine scales. This approach produces a coarse-scale problem with the computational structure of a continuous Galerkin method. The influence of fine-scale fluctuations is taken into account using an interscale transfer operator [9].

Another way to reduce the number of unknowns in DG methods is to enrich a continuous finite element space with discontinuous basis functions. The first author and his coworkers have shown that a continuous piecewise-linear (CG1) approximation can be stabilized by adding a discontinuous piecewise-constant (DG0) component [3]. The resultant scheme converges at the same rate as the piecewise-linear version of the DG method, while offering a considerable reduction in the number of unknowns.

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In this paper, we explore the possibility of enriching the CG1 space with piecewise-quadratic (DG2) functions which vanish at the vertices of the mesh but are allowed to be discontinuous across edges/faces of mesh cells. An a priori error estimate is derived for the CG1-DG2 discretization of a linear transport equation. Numerical studies confirm that the proposed method is stable and converges at the same rate as the fully discontinuous piecewise-quadratic version. At the same time, the continuity of the CG1 component reduces the number of degrees of freedom. Further attractive features and possible extensions are discussed at the end of this paper.

2. Variational problem. We consider the model problem

\[ \begin{align*}
\sigma u + \nabla \cdot (\beta u) &= f & \text{in } \Omega, \\
u &= u^D & \text{on } \partial \Omega^-,
\end{align*} \tag{2.1} \]

where $\Omega \subset \mathbb{R}^d$, $d = 1, 2$ is a bounded polygonal domain and $\partial \Omega^-$ is the inflow boundary of $\Omega$. The zero-order term in (2.1) might come either from an implicit time-stepping scheme or from the transformation $w = e^{\beta \cdot x} u$.

Let $\beta^- := \min\{0, \beta \cdot n_\Omega\}$ and $\beta^+ := \max\{0, \beta \cdot n_\Omega\}$, where $n_\Omega$ is the unit outward normal to $\partial \Omega$. The (sufficiently regular) solution $u$ to (2.1) satisfies the variational problem

\[ a(u, v) = l(v), \tag{2.2} \]

where

\[ a(u, v) := \int_\Omega \sigma uv + \int_\Omega \nabla \cdot (\beta u)v - \int_{\partial \Omega^-} \beta^- uv \tag{2.3} \]

and

\[ l(v) := \int_\Omega f v - \int_{\partial \Omega^-} \beta^- u^D v. \tag{2.4} \]

Integration by parts shows that

\[ a(u, v) = \int_\Omega \sigma uv - \int_\Omega u \partial_n \beta \cdot \nabla v + \int_{\partial \Omega^-} \beta^+ uv. \tag{2.5} \]

Therefore, by adding up (2.3) and (2.5) with $v = u$, we obtain

\[ a(u, u) = \int_\Omega \left( \sigma + \frac{1}{2} \nabla \cdot \beta \right) u^2 + \frac{1}{2} \int_{\partial \Omega^-} |\beta^-|^2 u^2. \tag{2.6} \]

We will throughout assume that for a constant $\sigma_0$

\[ \sigma + \frac{1}{2} \nabla \cdot \beta \geq \sigma_0 > 0, \tag{2.7} \]

such that the bilinear form is coercive on $L^2(\Omega)$. It is well known that the coercivity in $L^2(\Omega)$ is not enough in order to obtain a stable discretization, implying the necessity of including additional stabilization terms on the discrete level.

In addition to (2.7), we require boundedness

\[ \sup_{x \in \Omega} |\sigma(x) + \nabla \cdot \beta(x)| =: \sigma_1 < \infty. \tag{2.8} \]
3. Notation. Let \( \{T_h\} \) be a set of admissible simplicial meshes regular in the usual sense. The subscript \( h \) refers to the mesh size defined as a cell-wise constant function. A mesh \( T_h \) consists of cells \( K_h \) covering \( \Omega \), and the union of boundary sides \( S_h^b \) covers the domain boundary \( \partial \Omega \). The set of interior sides is denoted by \( S_h \). The diameter and measure of \( K \in K_h \) (or \( S \in S_h \)) are denoted by \( h_K \) (or \( |S| \)), respectively. The transformation from the reference cell \( K \) to \( K \) is denoted by \( T_K \).

We will throughout assume that the meshes are uniformly shape-regular. We denote by \( P^k(A) \) the restrictions of polynomials of total degree \( k \) to the set \( A \) and define the spaces of continuous finite elements.

Let \( X^k = P^k(\hat{K}) \). Then
\[
V_h^k := \left\{ v_h \in C(\hat{\Omega}) : v_h|_K \circ T_K^{-1} \in X^k \right\}.
\]

If \( S \in S_h^b \) lies on the boundary \( \partial \Omega \), we set \( n_S = n_\Omega \), the outward pointing unit normal vector. If \( S \in S_h \) is an interior side, \( n_S \) is a fixed unit vector normal to \( S \).

Let \( u \) be a piecewise continuous bounded function. For an interior side \( S \in S_h \) and \( x \in S \), we define
\[
\begin{align*}
    u_S^{in}(x) &:= \lim_{\varepsilon \searrow 0} v_h(x - \varepsilon n_S), & u_S^{ex}(x) &:= \lim_{\varepsilon \nearrow 0} v_h(x + \varepsilon n_S).
\end{align*}
\]

Next we define the jump and weighted mean for \( x \in S \) by
\[
[u](x) := u_S^{in}(x) - u_S^{ex}(x), \quad \{u\}(x) := \frac{1}{2} \left(u_S^{in}(x) + u_S^{ex}(x)\right).
\]

For a boundary side, we set \( u_S^{ex}(x) = 0 \), such that \([u]_S = u_S^{in}\) and \( \{u\}_S = \frac{1}{2} u_S^{in}\).

The jump of the product of two functions is given by
\[
[uv] = [u]\{v\} + \{u\}[v].
\] (3.2)

Then we have the integration by parts formula
\[
\int_{K_h} \nabla \cdot (\beta u) v = -\int_{K_h} u \beta \cdot \nabla v + \int_{S_h} \beta_n [uv] + \int_{S_h^b} \beta_n uv,
\] (3.3)

where we use the shorthand notation
\[
\int_{K_h} = \sum_{K \in K_h} \int_K, \quad \int_{S_h} = \sum_{S \in S_h} \int_S.
\]

Throughout, the positive part of a real number \( x \) is denoted by \( x^+ := \max\{0, x\} \) and the negative part by \( x^- = x - x^+ \). In addition, we define the positive and negative sign functions by
\[
\text{sgn}^+(x) := \begin{cases} 1 & x > 0 \\ 1/2 & x = 0 \\ 0 & x < 0 \end{cases}
\]

and \( \text{sgn}^-(x) := 1 - \text{sgn}^+(x) \). It follows that \( x^+ = \text{sgn}^+(x) x \), \( x^- = \text{sgn}^-(x) x \), such that \( \text{sgn}(x) = \text{sgn}^+(x) - \text{sgn}^-(x) \) and \( |x| = x^+ - x^- = x \text{sgn}(x) \).

We also use the notation \( x \lesssim y \) if there exists a constant \( C \) independent of \( h, \beta \), and \( \sigma \) such that \( x \leq C y \), and similarly for \( x \gtrsim y \).
4. The CG1-DG2 method. Since the continuous Galerkin method is unstable, we will enrich the $V_1^h$ space with discontinuous components $D_h$. A reduced discontinuous Galerkin method based on the addition of piecewise constants, $D_h = P_0^h$, was proposed and analyzed in [3].

In order to define a side-based discontinuous enrichment space, we introduce some further notation. For each $K \in K_h$, we denote by $S_K$ the vector space spanned by the products of the pairs of nodal basis functions associated to the sides of $K$. Let the $n_K$ nodes of $K$ be ordered in counter-clockwise sense and let side $i$ join nodes $i$ and $i + 1$ (modulo number nodes). Then we set $\psi^i_K := \phi^i_K \phi^{i+1}_K$ and define $D_K := \text{span}\{\psi^i_K : 1 \leq i \leq n_K\}$. In the present paper, we use the following finite element space:

$$V_h := V_1^h \oplus D_h, \quad D_h := \{v_h \in L^2(\Omega) : v_h|_{K} \in D_K\}. \quad (4.1)$$

This space contains the space of continuous piecewise-quadratics, since

$$V_h = V_2^h + D'_h \quad (4.2)$$

where $D'_h$ consists of one discontinuous bubble function per side.

Remark 4.1. The $D_h$ shape functions are quadratic polynomials vanishing at the vertices of mesh elements. Hence, a globally defined function $u_h \in V_h$ is continuous at the vertices but may have jumps across the sides.

The degrees of freedom of $V_h$ are illustrated in Fig. 4.1.

In order to introduce the upwind DG form, we define the weighted means

$$\begin{align*}
\{v_h\}^\beta_- &:= \text{sgn}^-((\beta_n)v_h^\text{in} + \text{sgn}^+(\beta_n)v_h^\text{ex}}, \\
\{v_h\}^\beta_+ &:= \text{sgn}^+(\beta_n)v_h^\text{in} + \text{sgn}^-(\beta_n)v_h^\text{ex}.
\end{align*} \quad (4.3)$$

We have

$$\begin{align*}
\{u_h\}^\beta_+ - \{u_h\}^\beta_- &= \text{sgn}^+(\beta_n)u_h^\text{in} + \text{sgn}^-((\beta_n)u_h^\text{ex} - \text{sgn}^-((\beta_n)u_h^\text{in} - \text{sgn}^+(\beta_n)u_h^\text{ex} \\
&= \text{sgn}((\beta_n)u_h^\text{in} - \text{sgn}((\beta_n)u_h^\text{ex} = \text{sgn}((\beta_n)[u_h].
\end{align*}$$

It follows that

$$\begin{align*}
\{u_h\}^\beta_+ + \{u_h\}^\beta_- &= 2\{u_h\}, \quad \{u_h\}^\beta_+ - \{u_h\}^\beta_- = \text{sgn}((\beta_n)[u_h]. \quad (4.4)
\end{align*}$$
The bilinear form for the combined CG-DG discretization is given by

\[ a_h(u_h, v_h) := \int_{\Omega} \sigma u_h v_h + \int_{\partial \Omega} \nabla \cdot (\beta u_h) v_h - \int_{S_h^2} \beta_n u_h v_h - \int_{S_h} \beta_n [u_h] v_h \beta_-. \quad (4.5) \]

Integration by parts shows that

**Lemma 4.2.** The bilinear form \( a_h \) can be written as:

\[ a_h(u_h, v_h) = \int_{\Omega} \sigma u_h v_h - \int_{\Omega} u_h \beta \cdot \nabla v_h + \int_{S_h^2} \beta_n^+ u_h v_h + \int_{S_h} \beta_n [u_h] \beta_+ [v_h]. \quad (4.6) \]

**Proof.** From (3.3) we have

\[
\int_{\Omega} \nabla \cdot (\beta u_h) v_h - \int_{S_h^2} \beta_n u_h v_h - \int_{S_h} \beta_n [u_h] \{v_h\} \beta_- = -\int_{\Omega} u_h \beta \cdot \nabla v_h + \int_{S_h} \beta_n ([u_h v_h] - [u_h] \{v_h\} \beta_-) + \int_{S_h^2} \beta_n^+ u_h v_h.
\]

It remains to invoke (4.4), which yields

\[ \{v_h\} - \{v_h\} \beta_- = \frac{1}{2} \text{sgn}(\beta_n)[v_h] = \{v_h\} \beta_+ - \{v_h\}, \quad (4.7) \]

such that

\[ [u_h v_h] - [u_h] \{v_h\} \beta_- = [u_h] \{v_h\} + \{u_h\} [v_h] - [u_h] \{v_h\} \beta_- = \frac{1}{2} \text{sgn}(\beta_n)[u_h][v_h] + \{u_h\} \{v_h\} = \{u_h\} \beta_+ + \{v_h\}. \]

Let us define the natural norm associated with the discontinuous formulation:

\[ \|u_h\|_{DG} := \sqrt{\sigma_0 \|u_h\|^2 + \frac{1}{2} \int_{\partial \Omega} |\beta_n| u_h^2 + \frac{1}{2} \int_{S_h} |\beta_n|[u_h]^2}. \quad (4.8) \]

From Lemma 4.2 we obtain the coercivity result

**Lemma 4.3.** The bilinear form \( a_h \) is coercive with respect to the DG-norm:

\[ a_h(u_h, u_h) \geq \|u_h\|^2_{DG}. \quad (4.9) \]

**Proof.** Adding up (4.5) and (4.6) yields

\[ 2a_h(u_h, v_h) = \int_{\Omega} (2\sigma + \nabla \cdot \beta) u_h^2 + \int_{\partial \Omega} |\beta_n| u_h^2 + \int_{S_h} \beta_n [u_h] (\{u_h\} \beta_+ - \{u_h\} \beta_-), \]

where we used the fact that \( x^+ - x^- = |x| \). It remains to apply (2.7). \( \Box \)

The result of Lemma 4.3 indicates that the DG-upwind formulation controls the space \( D_h \), but not the continuous part of the finite element space. We will therefore improve this stability result in the following section.

The DG-formulation applied to \( V_h \) has significant advantages over the classical DG-methods. First, the computationally expensive edge-integrals have only to be computed for the two functions of \( D_h \) associated to the edge. Furthermore, the coupling structure of the system matrix is considerably reduced. Based on the decomposition (4.2), we may even use static condensation in order to solve a system which has the same structure as the one from the Galerkin formulation with \( V_h \).
5. Stability. First we introduce some projection operators. By \( \pi^1_K \) we denote the \( L^2(K) \)-projection on \( P^1(K) \). Next we introduce the \( L^2 \)-projection on \( D_h \), defined by \( \int_\Omega (v - \pi^D_h v) w = 0 \) for all \( w \in D_h \). Since \( D_h \) is discontinuous, \( \pi^D_h \) is defined cell-wise and we may write \( \pi^D_h = \sum_K \pi^D_K \), where \( \pi^D_K \) is defined by \( \int_K (v - \pi^D_K v) w = 0 \) for all \( w \in D_K \).

Below, we will use the following technical results.

**Lemma 5.1.** For any \( g \in P^1(K) \) there exists a mesh-independent constant \( c_0 \) such that:

\[
\sup_{w \in D_K \setminus \{0\}} \frac{\int_K g w}{\|w\|_K} \geq c_0 \|g\|. \tag{5.1}
\]

**Proof.** One can check by straightforward computation that on the reference element the \( 3 \times 3 \) matrix \( M \) with \( M_{ij} := \int_{\hat{K}} \hat{\psi}_i \hat{\phi}_j \) has full rank. The result follows by transformation. \( \Box \)

By the shape-regularity of the meshes we have the standard inverse estimates stated as follows.

**Lemma 5.2.** There exists a mesh-independent constant \( C_I \) such that for all \( v_h \in V_h \) the following inverse estimates hold: For any \( K \in \mathcal{K}_h \) and any \( S \subset \partial K \)

\[
\|v_h\|_S \leq C_I h_K^{-1/2} \|v_h\|_K, \quad \|
abla v_h\|_K \leq C_I h_K^{-1} \|v_h\|_K. \tag{5.2}
\]

In order to obtain control over the weighted streamline derivatives we introduce the following semi-norm:

\[
\||u_h||_\beta := \sqrt{\sum_{K \in \mathcal{K}_h} \delta_K \|\pi^1_K \beta \cdot \nabla u_h\|^2_K}, \quad \delta_K = \frac{h_K}{\|\beta\|_{\infty,K}}. \tag{5.3}
\]

Next we show that projections on \( P^1(K) \) can be replaced by projections on \( D_K \).

**Lemma 5.3.** We have for all \( v_h \in V_h \)

\[
\sum_{K \in \mathcal{K}_h} \delta_K \|\pi^1_K \beta \cdot \nabla v_h\|^2_K \leq c_0^{-1} \sum_{K \in \mathcal{K}_h} \delta_K \|\pi^D_h \beta \cdot \nabla v_h\|^2_K. \tag{5.4}
\]

**Proof.** Let \( g := \pi^1 \beta \cdot \nabla v_h \in P^1(K) \). Then by Lemma 5.1 and the definition of \( \pi^D_K \) we have

\[
\|\pi^1 \beta \cdot \nabla v_h\| \leq c_0^{-1} \sup_{w \in D_K \setminus \{0\}} \frac{\int_K g w}{\|w\|_K} \leq c_0^{-1} \sup_{w \in D_K \setminus \{0\}} \frac{\int_K \pi^D_K g w}{\|w\|_K} \leq c_0^{-1} \|\pi^D_K \beta \cdot \nabla v_h\|.
\]

\( \Box \)

We now introduce the augmented DG norm

\[
\|\|u_h\|\| := \sqrt{\|u_h\|^2_{DG} + \|\|u_h\|\|^2_\beta}.
\]

The main result of this section is the following stability result.

**Lemma 5.4.** There exists a mesh-independent constant \( \gamma \) such that

\[
\sup_{v_h \in V_h \setminus \{0\}} \frac{a_h(u_h, v_h)}{\||v_h|||} \geq \gamma \|\|u_h\||. \tag{5.5}
\]
Proof. It is enough to prove that for given \( u_h \in V_h \) there exists \( v_h \in V_h \) such that

\[
a_h(u_h, v_h) \geq \gamma' \|v_h\|^2 \quad \text{and} \quad \|v_h\| \leq C\|u_h\|.
\] (5.6)

Then (5.5) follows with \( \gamma = \gamma' / C \). We have already seen that

\[
a_h(u_h, u_h) \geq \|u_h\|^2_{DG}.
\] (5.7)

Let \( w_h := \sum_{K} \delta_K \pi_h^D \beta \cdot \nabla u_h \in D_h \). Then we have

\[
a_h(u_h, w_h) = \int_{K_h} (\sigma + \nabla \cdot \beta) u_h w_h + \int_{K_h} \delta_K \beta \cdot \nabla u_h \pi_h^D \beta \cdot \nabla u_h
- \int_{S_h^b} \beta_n(u_h w_h) - \int_{S_h} \beta_n[u_h] \{ w_h \} \beta_n,
\]

such that with the definition of \( \pi_h^D \) and the Cauchy-Schwarz inequality

\[
\sum_{K \in K_h} \delta_K \| \pi_h^D \beta \cdot \nabla u_h \|_K^2 \leq a_h(u_h, w_h) + \sigma_1 \| u_h \| \| w_h \| + \| \beta_n \|_{1/2} \| u_h \|_{\partial\Omega} \| \beta_n \|_{1/2} \| w_h \|_{\partial\Omega}
+ \| \beta_n \|_{1/2} \| u_h \|_{S_h^b} \| \beta_n \|_{1/2} \| w_h \|_{S_h^b}. \] (5.8)

Due to the inverse estimate (5.2), we get

\[
\| \beta_n \|_{1/2} \| u_h \|_{\partial\Omega}^2 + \| \beta_n \|_{1/2} \| w_h \|_{S_h^b}^2 \leq C_1^2 \sum_{K \in K_h} h_K \| \beta \|_{\infty, K} \| w_h \|_K^2
\leq C_1^2 \sum_{K \in K_h} \delta_K \| \pi_h^D \beta \cdot \nabla u_h \|_K^2. \] (5.9)

It follows with Young’s inequality that

\[
\frac{1}{2} \sum_{K \in K_h} \delta_K \| \pi_h^D \beta \cdot \nabla u_h \|_K^2 \leq a_h(u_h, w_h) + C\| u_h \|^2_{DG}. \] (5.10)

Invoking Lemma 5.3 we obtain

\[
\frac{c_0}{2} \sum_{K \in K_h} \delta_K \| \pi_h^D \beta \cdot \nabla u_h \|_K^2 \leq a_h(u_h, w_h) + C\| u_h \|^2_{DG}. \] (5.11)

Therefore, choosing \( v_h := u_h + \varepsilon w_h \) with \( \varepsilon > 0 \) sufficiently small yields

\[
a_h(u_h, v_h) \geq (1 - \varepsilon C) \| u_h \|^2_{DG} + \frac{\varepsilon c_0}{2} \sum_{K \in K_h} \delta_K \| \pi_h^D \beta \cdot \nabla u_h \|_K^2
\geq \gamma' \| u_h \|^2
\]

with \( \gamma' = c_0 / (c_0 + 2C) \) for \( \varepsilon = 2 / (c_0 + 2C) \). By using again the inverse estimate (5.2) we can bound \( \| w_h \| \lesssim \| u_h \| \), which terminates the proof. \( \Box \)
6. A priori error analysis. In order to obtain a priori error estimates, we make use of the Clément operator for the space $V_h^2$. It satisfies the following interpolation error estimates.

**Lemma 6.1.** The Clément operator has the following local interpolation properties for $0 \leq l \leq 1$ and $1 \leq k \leq 3$

$$\| \nabla^l (u - I_h u) \|_K \lesssim h_k^{l-1} |u|_{k, \omega_K}, \quad \| u - I_h u \|_S \lesssim h_{k-2}^{1/2} |u|_{k, \omega_S},$$

(6.1)

where $\omega_K$ and $\omega_S$ denote the patches of cells around $K$ and $S$, respectively.

For the analysis of the consistency error, we introduce

$$C_h := \sup_{v_h \in \mathcal{V}_h \setminus \{0\}} \frac{a_h(I_h u, v_h) - l_h(v_h)}{\|v_h\|}.$$  

(6.2)

**Lemma 6.2.** Let $1 \leq k \leq 3$ and $u \in H^k(\Omega)$. Then the consistency error is bounded by

$$C_h \lesssim \sum_{K \in \mathcal{K}_h} h_k^{-1/2} \| u \|_{H^k(K)}.$$  

(6.3)

**Proof.** By the continuous equation, we have

$$l_h(v_h) = \int_{\partial S} f v_h - \int_{\partial \Omega} \beta_n u v_h = \int_{\Omega} (\sigma u + \nabla \cdot (\beta u)) v_h - \int_{\partial \Omega} \beta_n u v_h$$

$$= \int_{\Omega} \sigma u v_h - \int_{\Omega} u \beta \cdot \nabla v_h + \int_{\partial \Omega} \beta_n^+ u v_h + \int_{S_h} \beta_n \{u - I_h u\}_{\beta, +} [v_h]$$

By Lemma 4.2, it follows that

$$a_h(I_h u, v_h) - l_h(v_h) = \int_{\Omega} \sigma (I_h u - u) v_h + \int_{\Omega} (I_h u - u) \beta \cdot \nabla v_h$$

$$+ \int_{\partial \Omega} \beta_n^+ (I_h u - u) v_h + \int_{S_h} \beta_n \{u - I_h u\}_{\beta, +} [v_h].$$

Let us now examine the different terms. First we have

$$\int_{\Omega} \sigma (I_h u - u) v_h \leq \| I_h u - u \| \sigma_1 \|v_h\| \leq \frac{\sigma_1}{\sigma_0^{1/2}} h^k |u|_k \sigma_0^{1/2} \|v_h\|.$$  

(6.4)

Next

$$\int_{\partial \Omega} \beta_n^+ (I_h u - u) v_h + \int_{S_h} \beta_n \{u - I_h u\}_{\beta, +} [v_h]$$

$$\leq \int_{\partial \Omega} \|\beta_n\| \| I_h u - u \| \|v_h\| + \int_{S_h} \|\beta_n\| \| I_h u - u \[v_h]\|$$

$$\leq \|\beta_n\|^{1/2} (u - I_h u) \|\partial \Omega\| \|\beta_n\|^{1/2} v_h \|\partial \Omega\| + \sum_{S \in \mathcal{S}_h} \|\beta_n\|^{1/2} (u - I_h u) \|S\| \|\beta_n\|^{1/2} [v_h] \|S\|$$

$$\leq \|\beta\|^{1/2} h^{k-1/2} |u|_k \left( \|\beta_n\|^{1/2} v_h \|\partial \Omega\| + \left( \int_{S_h} \|\beta_n\| [v_h] \right)^{1/2} \right)^{1/2}.$$  

(6.5)
Finally we have
\[
\int_{\Omega} (I_h u - u) \beta \cdot \nabla v_h = \int_{\Omega} (I_h u - u) \pi^1 \beta \cdot \nabla v_h + \int_{\Omega} (I_h u - u)(\beta \cdot \nabla v_h - \pi^1 \beta \cdot \nabla v_h)
\]
\[
\leq \left( \sum_{K_h} \delta_K^{-1} ||I_h u - u||^2_K \right)^{1/2} \left( \sum_{K_h} \delta_K \|\pi^1 \beta \cdot \nabla v_h\|_{K}^2 \right)^{1/2} + A.
\]
(6.6)

The first term on the right-hand side of (6.6) is bounded by \(\|\beta\|_\infty h^{k-1/2} |u|_k\). For the second term we have
\[
A = \int_{\Omega} (I_h u - u)(\beta \cdot \nabla v_h - \pi^1 \beta \cdot \nabla v_h) \lesssim h^k |u|_k \|\beta \cdot \nabla v_h - \pi^1 \beta \cdot \nabla v_h\|.
\]

On each cell we have with \(\beta_K := \beta(x_K)\)
\[
\beta \cdot \nabla v_h - \pi^1_K (\beta \cdot \nabla v_h) = (\beta - \beta_K) \cdot \nabla v_h + \beta_K \cdot \nabla v_h - \pi^1_K (\beta \cdot \nabla v_h).
\]

Next we have \(\beta_K \cdot \nabla v_h = \pi^1_K (\beta_K \cdot \nabla v_h)\) such that with the stability of the projection and the inverse estimate
\[
||\beta \cdot \nabla v_h - \pi^1_K (\beta \cdot \nabla v_h)||_K = ||(I - \pi^1_K) (\beta - \beta_K) \cdot \nabla v_h)||_K
\]
\[
\leq ||\beta - \beta_K||_K ||\nabla v_h||_K \leq C_l ||\beta||_{1,K} ||v_h||_K.
\]

It follows that
\[
|A| \lesssim h^k |u|_k \|v_h\|.
\]

We conclude with Lemma 6.1. □

**Theorem 6.3.** Let \(h := \max_{K \in K_h} h_K\). Let \(1 \leq k \leq 3\) and \(u \in H^k(\Omega)\). Then we have the a priori error estimate
\[
||u - u_h|| \lesssim h^{k-1/2} ||u||_{H^k(\Omega)}.
\]
(6.7)

**Proof.** By stability and consistency we have
\[
||I_h u - u_h|| \lesssim \sup_{v_h \in V_h \setminus \{0\}} \frac{a_h(I_h u, v_h) - l_h(v_h)}{\|v_h\|} = C_h \lesssim h^{k-1/2} ||u||_{H^k(\Omega)}.
\]
It remains to check that the interpolation error allows for the same estimate:
\[
||u - I_h u|| \lesssim h^{k-1/2} ||u||_{H^k(\Omega)}.
\]
(6.8)

**Remark 6.4.** The error estimate in the pure \(L^2\)-norm is half an order suboptimal with respect to the interpolation error estimate, as usually found in the analysis of stabilized methods, see [11]. On the assumption of special meshes, we expect the full order to be recovered, as known from the analysis of [19].
7. Numerical results. In this section, we perform a grid convergence study for the proposed CG1-DG2 method and the following standard approximations:

- **CG1**: continuous Galerkin, linear elements;
- **CG2**: continuous Galerkin, quadratic elements;
- **DG1**: discontinuous Galerkin, linear elements;
- **DG2**: discontinuous Galerkin, quadratic elements.

We also study the convergence behavior of the streamline upwind Petrov-Galerkin (SUPG) method \([5]\) which stabilizes CG1/CG2 by using the modified test function

\[
\tilde{v}_h = v_h + \tau \beta \cdot \nabla v_h, \quad \tau = \frac{h}{2\|\beta\|_K}.
\]  

(7.1)

For a quantitative comparison of the results, we calculate the error norms

\[
E_2(h) = \|u - u_h\|_{L^2(\Omega)},
\]

(7.2)

\[
E_3(h) = \|u - u_h\|,
\]

(7.3)

where \(u\) is the exact solution and \(u_h\) is a finite element approximation. Furthermore, we estimate the expected order of convergence by the formula \([15]\)

\[
p_i = \log_2 \left( \frac{E_i(2h)}{E_i(h)} \right), \quad i = 1, 2.
\]  

(7.4)

The below numerical study for two-dimensional convection problems illustrates the convergence behavior of selected methods on triangular and quadrilateral meshes. We remark that the above analysis is only applicable to triangles.

7.1. Steady convection with a constant velocity field. In the first numerical example, we consider the steady convection equation

\[
u + \nabla \cdot (\beta u) = f \quad \text{in } \Omega = (0, 1) \times (0, 1)
\]

(7.5)

with the constant velocity field

\[
\beta(x, y) = (0.5, 1).
\]

(7.6)

The exact solution and inflow boundary conditions are defined by

\[
u(x, y) = \sin(2\pi x) \sin(2\pi y).
\]

(7.7)

The right-hand side of equation (7.5) is determined using the method of manufactured solutions, i.e., we use \(f := u + \nabla \cdot (\beta u)\), where \(u\) is given by (7.7).

Figure 7.1 shows the exact solution as well as the numerical results for continuous and discontinuous quadratic elements. The distribution of the absolute error \(|u - u_h|\) for CG1-DG2 and CG2 is displayed in Fig. 7.2. The stable CG1-DG2 discretization produces the largest errors around the local extrema, while CG2 gives rise to global pollution errors which have the same order of magnitude throughout the domain.

A grid convergence study for all methods is performed on uniform and perturbed meshes. The latter are constructed by adding random displacements to the positions of the inner nodes of the corresponding uniform mesh (see Fig. 7.3). The numbers of degrees of freedom (DOFs) for all methods under investigation are listed in Tables 7.1 and 7.2. The DOF numbers for CG2 / CG1-DG2 / DG2 differ by a factor of approximately 1.2/1.8 on quadrilateral meshes and 1.7 on triangular meshes.
In triangular elements, we use shape functions based on integrated Jacobi polynomials \[4, 18\]. In quadrilateral elements, we use tensor products of 1D Lobatto shape functions [18] for CG and CG1-DG2 and Legendre shape functions for DG. In the CG1-DG2 case we omit the bubble function in the middle of an element so that we
A combined CG1-DG2 method

uniform

perturbed

Fig. 7.3: Triangular meshes

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Table 7.1: DOF numbers on triangular meshes

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</table>

Table 7.2: DOF numbers on quadrilateral meshes

Fig. 7.4: CG1-DG2 serendipity elements with discontinuous edge basis functions (blue bullets) and continuous vertex basis functions (white circles)

get serendipity elements [1], which can be seen in Fig. 7.4.

Tables 7.3 - 7.6 show the $E_2$ and $E_3$ norms as well as the corresponding convergence rates. On uniform meshes, CG1, CG2, DG1 and CG1-SUPG exhibit the
same rate of convergence (2.00 for $E_2$ and 1.50 for $E_3$). The convergence rates for CG2-SUPG, CG1-DG2, and DG2 are also the same (3.00 for $E_2$ and 2.5 for $E_3$). On perturbed triangular meshes, only the CG1 convergence rates deteriorate significantly (1.35 for $E_2$ and 0.89 for $E_3$). The use of perturbed quadrilateral elements reduces the convergence rates for CG1-DG2 to those of CG2 (i.e., 2.00 for $E_2$ and 1.50 for $E_3$).

In the rest of this section, we restrict our numerical studies to triangular meshes.

### 7.2. Steady convection with a rotating velocity field.

Let us now consider equation (7.5) with the rotating velocity field

$$\beta(x, y) = (y, 1 - x).$$

(7.8)

The exact solution (Fig. 7.5) and inflow boundary conditions are given by

$$u(x, y) = \begin{cases} 
0.25 \left( 1 + \cos \left( \frac{x(r - 0.65)}{0.15} \right) \right), & \text{if } 0.5 < r = \sqrt{x^2 + y^2} < 0.8, \\
0.5, & \text{if } 0.2 < r < 0.4, \\
0, & \text{otherwise}.
\end{cases}$$

(7.9)

The numerical solutions and absolute errors are shown in Figs 7.5 and 7.6, respectively. The largest errors occur around discontinuities, where all methods (CG1-DG2, CG2, SUPG-CG2, DG2) produce under- and overshoots. The unstable CG2 method gives rise to spurious oscillations even in elements where the exact solution is smooth or zero. The CG1-DG2 error distribution is similar to those for CG2-SUPG and DG2, which proves that discontinuous quadratic components have a stabilizing effect.
A combined CG1-DG2 method

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Table 7.4: Steady convection / constant velocity, perturbed triangular mesh

Table 7.5: Steady convection / constant velocity, uniform quadrilateral mesh

Table 7.6: Steady convection / constant velocity, perturbed quadrilateral mesh
7.3. Unsteady convection with a constant velocity field. The third example is an unsteady version of the test problem from Section 7.1. We solve

\[
\frac{\partial u}{\partial t} + \nabla \cdot (\beta u) = 0 \quad \text{in } \Omega = (0, 1) \times (0, 1)
\]  

(7.10)
with the constant velocity field

$$\beta(x, y) = (1, 1).$$  \hspace{1cm} (7.11)

The initial condition is given by

$$u(x, y, 0) = \sin(2\pi x) \sin(2\pi y) \quad \text{in } \Omega, \hspace{1cm} (7.12)$$

and the inflow boundary conditions

$$u(0, y, t) = -\sin(2\pi t) \sin(2\pi (y - t)), \hspace{1cm} (7.13)$$
$$u(x, 0, t) = -\sin(2\pi t) \sin(2\pi (x - t)) \hspace{1cm} (7.14)$$

are chosen so that the initial data matches the exact solution at the time $t = 1.0$.

Table 7.7 shows that CG1-DG2 converges faster than CG2. The absolute errors for the two methods are shown in Fig. 7.7. As in the case of steady convection, CG2 produces global pollution errors, while the CG1-DG2 errors stay local and flow along the characteristics of (7.10) without generating numerical instabilities elsewhere.

### 7.4. Unsteady convection with a rotating velocity field

In the last example, we consider the solid body rotation problem [10, 15], a particularly challenging
test for numerical advection schemes. We solve (7.10) with the velocity field

\[ \beta(x, y) = (0.5 - y, x - 0.5) \]

which describes a counterclockwise rotation about the center of \( \Omega = (0, 1) \times (0, 1) \).

The exact solution to the solid body rotation problem reproduces the initial state \( u_0 \) exactly after each full revolution (\( t = 2\pi k, k \in \mathbb{N} \)). Hence, the challenge of this test is to preserve the shape of \( u_0 \). Following LeVeque [15], we rotate a slotted cylinder, a sharp cone, and a smooth hump. The initial condition is shown in Fig. 7.8.

The numerical solutions and absolute errors for CG1-DG2 and CG2 are compared in Figs 7.9 and 7.10. It can be seen that the CG2 solution is corrupted by global oscillations. CG1-DG2 produces small undershoots and overshoots in the neighborhood of the slotted cylinder but the hump and cone are preserved perfectly. This property of CG1-DG2 makes it an ideal base discretization for the design of a high-resolution finite element scheme in which monotonicity constraints are enforced using algebraic flux correction [14] for CG1 and hierarchical slope limiting [12, 13] for DG2.

8. Conclusions and outlook. This paper bridges the gap between continuous and discontinuous Galerkin methods for convection-dominated transport problems. The attractive features of the combined CG1-DG2 method include intrinsic stability, optimal convergence rates, lack of free parameters, and moderate computational cost compared to the fully discontinuous piecewise-quadratic approximation. A further gain of efficiency can be achieved using iterative methods or time-stepping methods in which the coarse-scale components are treated implicitly and their fine-scale
A combined CG1-DG2 method

Fig. 7.8: Solid body rotation, initial data and exact solution at $t = 2\pi$

Fig. 7.9: Solid body rotation problem, numerical solutions at $t = 2\pi$

counterparts explicitly. This solution strategy has already been shown to preserve unconditional stability in the context of DG methods [13]. Another promising approach is the use of explicit/implicit Runge-Kutta methods [6]. The extension of CG1-DG2 to convection-diffusion equations and design of limiting techniques for enforcing monotonicity constraints will be addressed in forthcoming publications. We also envisage the use of $hp$-adaptivity with higher-order discontinuous enrichments and/or reconstruction techniques. In summary, CG1-DG2 is just the simplest representative of multiscale finite element methods whose true potential is yet to be discovered.

Acknowledgements. The authors would like to thank Friedhelm Schieweck (University of Magdeburg) for inspiring discussions. This research was supported by the German Research Association (DFG) under grant KU 1530/6-2.
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