Optimal control of cracks in elastic bodies
with thin rigid inclusions

by

A. M. Khludnev & G. Leugering

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OPTIMAL CONTROL OF CRACKS IN ELASTIC BODIES
WITH THIN RIGID INCLUSIONS

A. M. Khludnev¹, G. Leugering²

¹Lavrentyev Institute of Hydrodynamics of the Russian Academy of Sciences, Novosibirsk 630090, Russia;
E-mail: khlud@hydro.nsc.ru

²Friedrich-Alexander-University Erlangen-Nuremberg, Institute of Applied Mathematics II, Martensstr. 3, 91058, Erlangen, Germany;
E-mail: leugering@am.uni-erlangen.de

Abstract
In this paper we consider the control of cracks in elastic bodies with rigid inclusion. We first describe the problem statement, provide an equivalent formulation as a variational inequality and prove existence and uniqueness of solutions. Furthermore, we consider this problem as a limiting problem when the elasticity parameters of the inclusion tend to infinity. Then we formulate the optimal control problems and derive an explicit formula for the crack sensitivity and for the energy release rate. We show existence of optimal solutions.

Keywords: Crack, rigid inclusion, non-penetration, variational inequality, control problem

1 Introduction

Energy release through crack propagation in brittle or ductile composite materials is a subject of active current research. Crashworthiness and resistance against damage and failure is the driving motivation. In recent research in material sciences one
intends to improve such properties using elastic or even rigid fibres and other inclusions. Also in flexible multi-structures configured from plates, shells and elastic bodies, the influence of mechanical substructures, such as attachments welded to the main body, on existing cracks or on their nucleation in the body is of great interest. In that context the most common mode of failure is interlaminar fracture or delamination. Therefore, influence of shapes, material properties and applied forces on crack-tip sensitivities is a challenging mathematical problem. This problem can be coined as crack-control. Some attempts have been made in the literature. See [1], [2], [20] for examples in mechanical engineering, where sensitivities are typically based on FEM-models and no proof are given for infinite dimensional case. A first attempt towards optimization of the shape of inclusion with respect to maximizing the energy release rate have been reported in [16, 17]. However, a rigorous mathematical treatment on the infinite dimensional level is still in its infancy.

The corresponding analysis strongly depends on the mathematical modeling of cracks. It is known that classical crack problems in elasticity theory are characterized by linear boundary conditions imposed at the crack faces. Such a linear approach allows the opposite crack faces to penetrate each other which leads to inconsistency from the practical standpoint. In recent years, a crack theory with non-penetration conditions has been under active study. This theory is characterized by inequality type boundary conditions at the crack faces. Since the contact set is unknown in this case, the problems considered can be viewed as free boundary problems. The book [3] contains results for crack models with the non-penetration conditions for a wide class of constitutive laws. In particular, 2D and 3D models as well as plate models with inequality type boundary conditions are analyzed. After publication of this book, new approaches and trends in the study of non-linear crack models
with the non-penetration have been developed. For example, a problem of differentiation of the energy functional with respect to crack perturbations is solved in a general setting, smooth and fictitious domain methods are proposed, invariant integrals are constructed for different geometrical situations, etc. We refer the reader to publications [4], [12], [18], [19].

To describe composite materials it is necessary to analyze mathematical models of elastic bodies with rigid inclusions and cracks. In such a case, new types of boundary value problems and boundary conditions appear. Rigid inclusions may be delaminated, hence the crack approach with non-penetration conditions is to be used. We refer the reader to the publications [5], [6], [7] where suitable problem formulations can be found. As for practical examples of composite materials with inclusions we can mention the book [13].

There are other problems formulated in non-smooth domains with inequality type boundary conditions. It is known that contact problems for bodies of different dimensions are described by models similar to those of crack models with the non-penetration. For these contact problems, the equilibrium equations are satisfied in cracked domains, and the boundary conditions are of inequality type [8], [15], [14], [21]. Moreover, it turns out that many practical problems should be described by crack models with overlapping domains. This overlapping approach is applicable, for example, for description of the slipping phenomenon of ice plates, for considering the subduction phenomenon of tectonic plates, and as well as for the construction of complicated precise level devices in engineering practice, etc. Considering suitable structures, we, actually, introduce Riemann surfaces with two or more sheets having cracks with inequality type boundary conditions at the crack faces, see [9], [10].

In the present paper, we consider a 2-layer structure with elastic and rigid parts. A glue condition imposed at a given line
which is joint for both elastic and rigid parts. Moreover, a delamination takes place along this line, and we impose inequality type boundary conditions to describe a mutual non-penetration between the crack faces. Crack faces are assumed to be friction free. We prove existence of solutions and analyze two optimal control problems with different cost functionals. The first cost functional describes a crack opening, and the second one provides a formula for the derivative of the energy functional with respect to the crack length. Thus the second functional characterizes a possibility for a crack propagation provided that we use the Griffith rupture criterion. External forces acting on the rigid part are considered as control functions.

General approaches and results for free boundary problems in the case of optimal control in elasticity theory can be found in [11].

2 Problem formulation

Let $\Omega \subset \mathbb{R}^2$, $\omega \subset \mathbb{R}^2$ be bounded domains with smooth boundaries $\Gamma, \partial \omega$, respectively. We assume that $\gamma \subset \Omega$ is a smooth curve without self-intersections such that $\bar{\gamma} \subset \Omega, \gamma \subset \partial \omega$. Denote by $\nu = (\nu_1, \nu_2)$ a unit normal external vector to $\partial \omega$; $\Omega_{\gamma} = \Omega \setminus \bar{\gamma}$, see Fig. 1. In our considerations, the domain $\Omega$ corresponds to an elastic body, and $\omega$ fits to a rigid body.

In what follows we assume that there is a glue condition along $\gamma^-$ which means that displacements of the elastic and rigid bodies coincide at $\gamma^-$. The signs $\pm$ fit to the positive and negative faces of $\gamma$ with respect to $\nu$. Moreover, displacements of the elastic body found at $\gamma^+, \gamma^-$ are different. This corresponds to the presence of a cut (crack) in the domain $\Omega$ along $\gamma$. We impose nonlinear boundary conditions at the crack faces providing a mutual non-penetration between $\gamma^\pm$. By using the term ”rigid” body we have in mind that a displacement field in $\omega$ has a special
structure. To give more details, we introduce the space $R(\omega)$ of infinitesimal rigid displacements

$$R(\omega) = \{ \rho = (\rho_1, \rho_2) \mid \rho(x) = Dx + C, \ x \in \omega \}$$

with

$$D = \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}, \ C = (c^1, c^2); \ d, c^1, c^2 = \text{const.}$$

In such a case, the displacements found in $\omega$ should belong to the space $R(\omega)$. Since displacements of the elastic and rigid bodies coincide at $\gamma^-$, the face $\gamma^-$ can be considered as a thin rigid inclusion with a delamination in the elastic body $\Omega_\gamma$.

It is possible to consider $R(\omega)$ as a subspace of $H^1(\omega)^2$ with the norm

$$\|\rho\|^2_{H^1(\omega)^2} = \int_\gamma \rho^2, \ \text{meas} \gamma > 0. \quad (1)$$
Indeed, the norm in the space $H^1(\omega)^2$ can be taken as [22]

$$
\|v\|_{H^1(\omega)^2}^2 = \int_\omega \varepsilon_{ij}(v)\varepsilon_{ij}(v) + \int_\gamma v^2, \quad \text{meas}\gamma > 0,
\tag{2}
$$

where $\varepsilon_{ij}(v) = \frac{1}{2}(v_{i,j} + v_{j,i})$, and $\varepsilon_{ij}(\rho) = 0$ in $\omega$, $i, j = 1, 2$, for $\rho \in R(\omega)$.

Summation convention over repeated indices is used; all functions with two lower indices are assumed to be symmetric in those indices.

In order to provide a problem formulation describing an equilibrium state for the described structure with rigid and elastic parts we introduce the Sobolev spaces

$$
H^1_1(\Omega_\gamma) = \{v \in H^1(\Omega_\gamma) \mid v = 0 \text{ on } \Gamma\},
$$

$$
H = H^1_1(\Omega_\gamma)^2 \times H^1(\omega)^2,
$$

and the set $K$ of admissible displacements

$$
K = \{(v, \rho) \in H \mid [v] \nu \geq 0 \text{ on } \gamma; \ v|_\gamma^- = \rho, \ \rho \in R(\omega)\}.
$$

Here $[v] = v^+ - v^-$ is a jump of $v$ across $\gamma$. The problem formulation is as follows. We have to find functions $u = (u_1, u_2), \rho_0 \in R(\omega), \sigma = \{\sigma_{ij}\}, i, j = 1, 2$, such that

$$
-\text{div}\sigma = f \quad \text{in } \Omega_\gamma, \tag{3}
$$

$$
\sigma - A\varepsilon(u) = 0 \quad \text{in } \Omega_\gamma, \tag{4}
$$

$$
u = 0 \quad \text{on } \Gamma, \tag{5}
$$

$$
\sigma_{ij} = \rho_0, \quad [u] \nu \geq 0 \quad \text{on } \gamma, \tag{6}
$$

$$
\int_\gamma [\sigma \nu \cdot u] + \int_\omega g\rho_0 = 0, \tag{7}
$$

$$
\int_\gamma [\sigma \nu \cdot v] + \int_\omega g\rho \leq 0 \quad \forall (v, \rho) \in K. \tag{8}
$$
Here $u = (u_1, u_2)$ is a displacement field in $\Omega$, $\sigma = \{\sigma_{ij}\}$ is the stress tensor; $\varepsilon(u) = \{\varepsilon_{ij}(u)\}$ is the strain tensor; $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$, $i, j = 1, 2$; $f = (f_1, f_2) \in L^2(\Omega)$, $g = (g_1, g_2) \in \tilde{L}^2(\omega)$ are given external forces acting on the elastic and rigid parts, respectively, and $A = \{a_{ijkl}\}, i, j, k, l = 1, 2$, is a given elasticity tensor with usual properties of symmetry and positive definiteness,

\[
a_{ijkl} = a_{jikl} = a_{klij}, \quad i, j, k, l = 1, 2, \quad a_{ijkl} \in L^\infty(\Omega);
\]

\[
a_{ijkl}\xi_{ij}\xi_{kl} \geq c_0|\xi|^2 \quad \forall \xi_{ji} = \xi_{ij}, \quad c_0 = const > 0.
\]

Relation (3) is the equilibrium equation, and (4) represents the Hooke’s law; $\sigma = \sigma(u)$ is defined from (4); $\sigma \nu = (\sigma_{1j}\nu_j, \sigma_{2j}\nu_j)$.

In order to prove existence of solutions to problem (3)-(8), we introduce a variational formulation which is equivalent to (3)-(8). To this end, consider the minimization problem

\[
\inf_{(v, \rho) \in K} \Pi(v, \rho),
\]

where

\[
\Pi(v, \rho) = \frac{1}{2} \int_{\Omega} \sigma(v)\varepsilon(v) - \int_{\Omega} f v - \int_{\omega} g \rho, \quad (v, \rho) \in H.
\]

For simplicity, we write $\sigma(v)\varepsilon(v)$ instead of $\sigma_{ij}(v)\varepsilon_{ij}(v)$, and $fv$ instead of $f_i v_i$. It is clear that the set $K$ is weakly closed in $H$, and the functional $\Pi$ is weakly lower semicontinuous on $H$. In order to verify that the problem (9) has a solution, we should check a coercivity of $\Pi$ on the set $K$. By (1), we have

\[
| \int_{\Omega} fv + \int_{\omega} g \rho | \leq c_1 \|v\|_{H^1_0(\Omega)}^2 + c_2 \|\rho\|_{L^2(\gamma)}^2.
\]
Hence, due to the Korn’s inequality,

\[ \Pi(v, \rho) \geq c \|v\|_{H^1_1(\Omega)}^2 - c_1 \|v\|_{H^1_1(\Omega)}^2 - c_2 \|\rho\|_{L^2(\gamma)}^2. \] (10)

Invoking the imbedding theorem, there exists a constant \(c_0\) such that

\[ \frac{c}{2} \|v\|_{H^1_1(\Omega)}^2 \geq c_0 \|v^−\|_{L^2(\gamma)}^2. \] (11)

Note that \(v^− = \rho\) on \(\gamma\), hence from (10)-(11) it follows

\[ \Pi(v, \rho) \geq \frac{c}{2} \|v\|_{H^1_1(\Omega)}^2 + c_0 \|\rho\|_{L^2(\gamma)}^2 - c_1 \|v\|_{H^1_1(\Omega)}^2 - c_2 \|\rho\|_{L^2(\gamma)}^2. \] (12)

Consequently, we conclude \(\Pi(v, \rho) \to +\infty\) as soon as

\[ \|v\|_{H^1_1(\Omega)}^2 + \|\rho\|_{L^2(\gamma)}^2 \to \infty, \quad (v, \rho) \in K, \]

thus, on the set \(K\) a coercivity of \(\Pi\) follows. Hence, we obtain existence of a solution to the problem (9). This solution \((u, \rho_0)\) satisfies the variational inequality

\[ (u, \rho_0) \in K, \] (13)

\[ \int_{\Omega_\gamma} \sigma(u)\varepsilon (v - u) - \int f(v - u) - \int g(\rho - \rho_0) \geq 0 \quad \forall (v, \rho) \in K. \] (14)

Let us prove uniqueness of the solution to (13), (14). Assuming the existence of two solutions \((u^1, \rho^1_0)\), \((u^2, \rho^2_0)\), we have the following inequalities

\[ \int_{\Omega_\gamma} \sigma(u^1)\varepsilon (u^2 - u^1) - \int f(u^2 - u^1) - \int g(\rho^2_0 - \rho^1_0) \geq 0, \]
\[
\int_{\Omega_\gamma} \sigma(u^2) \varepsilon(u^1 - u^2) - \int_{\Omega_\gamma} f(u^1 - u^2) - \int_{\omega} g(\rho_0^1 - \rho_0^2) \geq 0.
\]

Thus,
\[
\int_{\Omega_\gamma} \sigma(u^1 - u^2) \varepsilon(u^1 - u^2) \leq 0.
\]

Consequently, it follows \( u^1 = u^2 \) in \( \Omega_\gamma \). Hence, we conclude \( u^1 = u^2 \) on \( \gamma^- \), i.e. \( \rho_0^1 = \rho_0^2 \) on \( \gamma^- \), and we get \( \rho_0^1 = \rho_0^2 \) on \( \omega \).

Now we plan to prove the equivalence of the problem formulations (3)-(8) and (13)-(14) provided that the solutions are quite smooth. First of all we substitute \((v, \rho) = (u \pm \varphi, \rho_0)\) as test functions in (14), where \( \varphi \in C_0^\infty(\Omega_\gamma)^2 \). This implies
\[
\int_{\Omega_\gamma} \sigma(u) \varepsilon(\varphi) = \int_{\Omega_\gamma} f \varphi,
\]

hence, the equilibrium equations (3) hold. Now take \((v, \rho) = (0, 0), (v, \rho) = 2(u, \rho_0)\) in (14) as test functions. This implies

\[
\int_{\Omega_\gamma} \sigma(u) \varepsilon(u) - \int_{\Omega_\gamma} f u - \int_{\omega} g \rho_0 = 0. \quad (15)
\]

Integrating by parts here we arrive at (7).

By (15), it follows from (14) that
\[
\int_{\Omega_\gamma} \sigma(u) \varepsilon(v) - \int_{\Omega_\gamma} f v - \int_{\omega} g \rho \geq 0 \; \forall (v, \rho) \in K,
\]

from what we arrive at (8). Thus, we have proved that (13)-(14) imply (3)-(8). Conversely, we check that (13)-(14) follow
from (3)-(8). Multiply (3) by $v - u$ and integrate over $\Omega_\gamma$, where $v \in K$. This provides

$$
\int_{\Omega_\gamma} \sigma(u) \varepsilon(v - u) - \int_{\Omega_\gamma} f(v - u) = - \int_\gamma [\sigma \nu \cdot (v - u)]. \quad (16)
$$

On the other hand, relations (7)-(8) yield

$$
- \int_\gamma [\sigma \nu \cdot (v - u)] \geq \int_\omega g(\rho - \rho_0),
$$

and we arrive at the variational inequality (14).

3 Asymptotic analysis

In this section we prove that the problem (3)-(8) can be viewed as a limit problem for a family of elasticity problems. Instead of the rigid part $\omega$ it is possible to consider an elastic body occupying the domain $\omega$. Assume that a rigidity parameter of this elastic part $\omega$ tends to infinity. Then, it turns out, that the limit problem coincides with (3)-(8). In order to analyze this passage to the limit, we consider the following problems depending on a parameter $\delta > 0$. We have to find functions $u^\delta = (u_1^\delta, u_2^\delta), w^\delta = (w_1^\delta, w_2^\delta), \sigma^\delta = \{\sigma_{ij}^\delta\}, p^\delta = \{p_{ij}^\delta\}, i, j = 1, 2,$
such that

\[ -\text{div}\sigma^\delta = f \quad \text{in} \quad \Omega_\gamma, \quad (17) \]

\[ \sigma^\delta - A\varepsilon(u^\delta) = 0 \quad \text{in} \quad \Omega_\gamma, \quad (18) \]

\[ -\text{div}p^\delta = g \quad \text{in} \quad \omega, \quad (19) \]

\[ p^\delta - \frac{1}{\delta} B\varepsilon(w^\delta) = 0 \quad \text{in} \quad \omega, \quad (20) \]

\[ u^\delta = 0 \quad \text{on} \quad \Gamma; \quad p^\delta \nu = 0 \quad \text{on} \quad \partial\omega \setminus \bar{\gamma}, \quad (21) \]

\[ u^\delta^- = w^\delta, \quad [u^\delta]\nu \geq 0 \quad \text{on} \quad \gamma, \quad (22) \]

\[ \int_{\gamma}[\sigma^\delta\nu \cdot u^\delta] + \int_{\gamma}p^\delta\nu \cdot w^\delta = 0, \quad (23) \]

\[ \int_{\gamma}[\sigma^\delta\nu \cdot \bar{u}] + \int_{\gamma}p^\delta\nu \cdot \bar{w} \leq 0 \quad \forall (\bar{u}, \bar{w}) \in K^*. \quad (24) \]

Here \( u^\delta, w^\delta \) are displacements of the elastic bodies \( \Omega_\gamma, \omega \), respectively; \( \sigma^\delta, p^\delta \) are stress tensors; \( B = \{b_{ijkl}\}, i, j, k, l = 1, 2 \), is a given elasticity tensor with the usual properties of symmetry and positive definiteness, \( b_{ijkl} \in L^\infty(\omega) \). The set \( K^* \) is the set of admissible displacements

\[ K^* = \{(u, w) \in H \mid [u]\nu \geq 0, \ u^- = w \ \text{on} \ \gamma\}. \]

It is possible to prove solvability of the problem (17)-(24). To this end we consider the minimization problem

\[ \inf_{(u, w) \in K^*} \pi_\delta(u, w), \quad (25) \]

where

\[ \pi_\delta(u, w) = \frac{1}{2} \int_{\Omega_\gamma} \sigma(u)\varepsilon(u) - \int_{\Omega_\gamma} fu + \frac{1}{2} \int_{\omega} p^\delta(w)\varepsilon(w) - \int_{\omega} gw. \]
The functional $\pi_{\delta}$ is weakly lower semicontinuous on the space $H$ for any fixed $\delta$. Let us prove its coercivity on the set $K^*$. We have

$$\pi_{\delta}(u, w) = \frac{1}{2} \int_{\Omega_{\gamma}} \sigma(u)\varepsilon(u) - \alpha \int_{\gamma} w^2 - \int_{\Omega_{\gamma}} f u + \frac{1}{2} \int_{\omega} p^\delta(w)\varepsilon(w) + \alpha \int_{\gamma} w^2 - \int_{\omega} g w$$

with a positive parameter $\alpha$. Since $u = w$ on $\gamma^-$ for $(u, w) \in K^*$, the following inequality holds for small $\alpha$,

$$\frac{1}{2} \int_{\Omega_{\gamma}} \sigma(u)\varepsilon(u) - \alpha \int_{\gamma} w^2 \geq \frac{1}{4} \int_{\Omega_{\gamma}} \sigma(u)\varepsilon(u).$$

The norm

$$\|w\|_{H^1(\omega)^2}^2 = \frac{1}{2} \int_{\omega} p^\delta(w)\varepsilon(w) + \alpha \int_{\gamma} w^2$$

is equivalent to the norm (2), hence

$$\pi_{\delta}(u, w) \geq c_3 \|u\|_{H^1(\Omega_{\gamma})^2}^2 - c_4 \|u\|_{H^1(\Omega_{\gamma})^2} + c_5 \|w\|_{H^1(\omega)^2}^2 - c_6 \|w\|_{H^1(\omega)^2}$$

with positive constants $c_3, c_4, c_5, c_6$ possibly depending on $\alpha, \delta$. From (26) it follows that

$$\pi_{\delta}(u, w) \to +\infty, \text{ as } \|(u, w)\|_H \to \infty, \ (u, w) \in K^*.$$

Thus, we conclude that the problem (25) for a fixed $\delta$ has a solution satisfying the variational inequality

$$(u^\delta, w^\delta) \in K^*,$$
\[
\int_{\Omega_\gamma} \sigma(u^\delta) \varepsilon(\bar{u} - u^\delta) - \int_{\Omega_\gamma} f(\bar{u} - u^\delta) + \int_{\omega} p^\delta(w^\delta) \varepsilon(\bar{w} - w^\delta) - \int_{\omega} g(\bar{w} - w^\delta) \geq 0 \quad \forall (\bar{u}, \bar{w}) \in K^*.
\]

It is possible to see that problem formulations (17)-(24) and (27)-(28) are equivalent for smooth solutions. Hence, we have proved the existence of solutions to problem (17)-(24).

Now we prove that the solutions \((u^\delta, w^\delta)\) of the problem (27)-(28) converge to the solution of the problem (13)-(14) as \(\delta \to 0\). Denote \(p(w^\delta) = B \varepsilon(w^\delta)\). From (28) it follows

\[
\int_{\Omega_\gamma} \sigma(u^\delta) \varepsilon(u^\delta) - \int_{\Omega_\gamma} f u^\delta + \int_{\omega} p(w^\delta) \varepsilon(w^\delta) - \int_{\omega} g w^\delta = 0.
\]

Arguing like in the proof of the coercivity of the functional \(\pi_\delta\) from (29), we derive the following estimates

\[
\|u^\delta\|_{H_\Gamma^1(\Omega_\gamma)}^2 + \|w^\delta\|_{H_\Gamma^1(\omega)}^2 \leq c,
\]

\[
\int_{\omega} p(w^\delta) \varepsilon(w^\delta) \leq c \delta,
\]

being uniform with respect to \(\delta, \delta \in (0, \delta_0)\). Choosing a subsequence, if necessary, we assume that as \(\delta \to 0\),

\[
u^\delta \to u \text{ weakly in } H_\Gamma^1(\Omega_\gamma)^2,
\]

\[
w^\delta \to \rho_0 \text{ weakly in } H^1(\omega)^2, \rho_0 \in R(\omega).
\]
Now we substitute \((\bar{u}, \rho) \in K\) in (28) as a test function having in mind that \(K \subset K^*\). This implies

\[
\int_{\Omega} \sigma(u^\delta)\varepsilon(\bar{u}) - \int_{\Omega} f(\bar{u} - u^\delta) - \int_{\omega} g(\rho - w^\delta) \geq \int_{\Omega} \sigma(u^\delta)\varepsilon(u^\delta) + \int_{\omega} p^\delta(w^\delta)\varepsilon(w^\delta).
\] (32)

By (30), (31), we can pass to the lower limit as \(\delta \to 0\) in both sides of (32) and get the variational inequality

\[(u, \rho_0) \in K,
\]

\[
\int_{\Omega} \sigma(u)\varepsilon(\bar{u} - u) - \int_{\Omega} f(\bar{u} - u) - \int_{\omega} g(\rho - \rho_0) \geq 0 \ \forall(\bar{u}, \rho) \in K
\]

coinciding with (13), (14). Hence, we have proved that the limit problem for (17)-(24) as \(\delta \to 0\) coincides with (3)-(8).

4 Optimal control of opening the crack

Let \(G \subset L^2(\omega)^2\) be a bounded and weakly closed set. For any fixed \(g \in G\) it is possible to find a solution \(u = u(g), \rho_0 = \rho_0(g)\) of the problem (3)-(8). This solution is unique and satisfies the following variational inequality

\[(u(g), \rho_0(g)) \in K, \quad (33)\]

\[
\int_{\Omega} \sigma(u(g))\varepsilon(v - u(g)) - \int_{\Omega} f(v - u(g)) - \int_{\omega} g(\rho - \rho_0(g)) \geq 0 \ \forall(v, \rho) \in K. \quad (34)
\]
Thus, we can define the cost functional

$$J(g) = \int_{\gamma} [u(g)] \nu$$

characterizing an opening of the crack $\gamma$. Due to the definition of the set $K$ we see that $J(g) \geq 0$, $g \in G$. Consider the optimal control problem

$$\inf_{g \in G} J(g). \quad (35)$$

In what follows we prove that the problem (35) is solvable.

**Theorem 4.1** There exists a solution of the optimal control problem (35).

**Proof** Let $g^n \in G$ be a minimizing sequence. Denote $u^n = u(g^n)$, $\rho^n_0 = \rho_0(g^n)$. Then $(u^n, \rho^n_0)$ is the unique solution of the variational inequality

$$\begin{align*}
(u^n, \rho^n_0) & \in K, \\
\int_{\Omega_\gamma} \sigma(u^n) \varepsilon(v - u^n) - \int_{\Omega_\gamma} f(v - u^n) - \\
\int_{\omega} g^n(\rho - \rho^n_0) & \geq 0 \quad \forall (v, \rho) \in K.
\end{align*} \quad (36) \quad (37)$$

Since $G$ is a bounded set we have uniformly in $n$,

$$\|g^n\|_{L^2(\omega)²} \leq c \quad \quad (38)$$

From (37) it follows

$$\begin{align*}
\int_{\Omega_\gamma} \sigma(u^n) \varepsilon(u^n) - \int_{\Omega_\gamma} fu^n - \int_{\omega} g^n \rho^n_0 = 0.
\end{align*}$$
Hence, the following estimate holds
\[ \|u^n\|_{H^1(\Omega^\gamma)}^2 \leq c + \|\rho_0^n\|_{H^1(\omega)}^2. \] (39)
Meanwhile \( u^n = \rho_0^n \) on \( \gamma^- \), and
\[ \|\rho_0^n\|_{H^1(\omega)}^2 = \int_\gamma (\rho_0^n)^2. \]
Moreover, there exists a constant \( c > 0 \) such that
\[ \|(u^n)^-\|_{L^2(\gamma)^2} \leq c\|u^n\|_{H^1(\Omega^\gamma)}^2. \]
Therefore, (39) implies uniformly in \( n \),
\[ \|u^n\|_{H^1(\Omega^\gamma)}^2 \leq c. \] (40)
Since \( u^n = \rho_0^n \) on \( \gamma^- \), we obtain
\[ \rho_0^n \text{ are bounded in } H^1(\omega)^2. \] (41)
Choosing a subsequence, by (38), (40), (41), we can assume that as \( n \to \infty \)
\[ u^n \to u \text{ weakly in } H^1_0(\Omega^\gamma)^2, \]
\[ \rho_0^n \to \rho_0 \text{ strongly in } L^2(\omega)^2, \] (42)
\[ g^n \to g_0 \text{ weakly in } L^2(\omega)^2, \ g_0 \in G. \]
Passing to the limit as \( n \to \infty \) in (36), (37) we conclude that the limiting functions \( u, \rho_0, g_0 \) satisfy the variational inequality
\[ (u, \rho_0) \in K, \]
\[ \int_{\Omega^\gamma} \sigma(u)\varepsilon(v-u) - \int_{\Omega^\gamma} f(v-u) - \int_{\omega} g_0(\rho-\rho_0) \geq 0 \ \forall (v, \rho) \in K, \]
i.e. \( u = u(g_0), \rho = \rho_0(g_0) \). Due to the imbedding theorems, in addition to the convergences (42), we assume as \( n \to \infty \)

\[
[u^n] \nu \to [u(g_0)] \nu \text{ strongly in } L^1(\gamma)^2.
\]

This provides

\[
\inf_{g \in G} J(g) = \lim \inf J(g^n) = \lim \inf \int_{\gamma} [u^n] \nu = \\
\lim \int_{\gamma} [u^n] \nu = \int_{\gamma} [u(g_0)] \nu = J(g_0) = \inf_{g \in G} J(g).
\]

Hence, \( g_0 \) is a solution of the optimal control problem (35). Theorem 4.1 is proved.

5 Optimal choice of safe loading

This section is concerned with the optimal control of crack growth. We assume that the elastic body \( \Omega_\gamma \) has a crack (cut) \( \Gamma_c \) crossing the thin rigid inclusion \( \gamma \). The non-penetration condition is assumed to be fulfilled on \( \Gamma_c \). For simplicity, delamination is not assumed on \( \gamma \), and \( \Gamma_c = \{0\} \times (0, 1) \). The unit normal vector to \( \Gamma_c \) is denoted by \( n = (1, 0) \). Also denote \( \Omega^c_\gamma = \Omega \setminus (\bar{\Gamma}_c \cup \bar{\gamma}) \), \( \Omega^c = \Omega \setminus \bar{\Gamma}_c \), see Fig. 2.

Applying different forces \( g \) to the rigid body \( \omega \) we obtain different situations from the standpoint of the Griffith criterion. Recall that the Griffith criterion characterizes stable and unstable behavior of cracks in terms of derivatives of energy functionals with respect to the crack length. Namely, if the derivative reaches a critical value (a given material parameter) then the crack propagates. Otherwise, the crack is stable. In this section we analyze an optimal control problem with external forces \( g \) being control functions. For any given functions \( g \) we find the derivative of the energy functional with respect to the crack
length, and our aim is to maximize this derivative over a set of all admissible forces.

First of all we formulate the equilibrium problem for the case considered. In what follows, the assumptions $f \in C^1(\bar{\Omega})^2$, $a_{ijkl} \in C^1(\bar{\Omega})$, $i, j, k, l = 1, 2$, are used. As before we assume that $g \in L^2(\omega)^2$. It is necessary to find functions $u = (u_1, u_2), \rho_0 \in R(\omega), \sigma = \{\sigma_{ij}\}, i, j = 1, 2$, such that

$$-\text{div}\sigma = f \quad \text{in} \quad \Omega^c, \quad (43)$$

$$\sigma - A\varepsilon(u) = 0 \quad \text{in} \quad \Omega^c, \quad (44)$$

$$u = 0 \quad \text{on} \quad \Gamma; \quad u = \rho_0 \quad \text{on} \quad \gamma, \quad (45)$$

$$[u]n \geq 0 \quad \text{on} \quad \Gamma_c, \quad (46)$$

$$\int_{\Gamma_c} [\sigma n \cdot u] + \int_\gamma [\sigma v]\rho_0 + \int_\omega g\rho_0 = 0, \quad (47)$$

$$\int_{\Gamma_c} [\sigma n \cdot \bar{u}] + \int_\gamma [\sigma v]\bar{\rho} + \int_\omega g\bar{\rho} \leq 0 \quad \forall (\bar{u}, \bar{\rho}) \in K_0, \quad (48)$$
where the set of admissible displacements is defined as follows

\[ K_0 = \{(v, \rho) \in H_0 \mid [v]n \geq 0 \text{ on } \Gamma_c; \ v|_{\gamma} = \rho, \ \rho \in R(\omega)\}, \]

\[ H_0 = H_1(\Omega^c)^2 \times H^1(\omega)^2. \]

We see that the non-penetration condition (46) is imposed on \( \Gamma_c \), and no delamination (thus no crack) is assumed on \( \gamma \).

We shall prove existence of solutions to the problem (43)-(48). To this end the following minimization problem is considered

\[
\inf_{(v, \rho) \in K_0} \left\{ \frac{1}{2} \int_{\Omega^c} \sigma(v) \varepsilon(v) - \int_{\Omega^c} fv - \int_{\omega} g\rho \right\}. \tag{49}
\]

At this step the functions \( f, g \) are fixed. Since the functional in (49) is weakly lower semicontinuous on \( H_0 \) being coercive on \( K_0 \), we conclude that the problem (49) has a solution \((u, \rho_0)\) satisfying the variational inequality

\[
(u, \rho_0) \in K_0, \int_{\Omega^c} \sigma(u) \varepsilon(v - u) - \int_{\Omega^c} f(v - u) - \int_{\omega} g(\rho - \rho_0) \geq 0 \ \forall (v, \rho) \in K_0. \tag{50}
\]

Now we prove that the problem formulations (43)-(48) and (50)-(51) are equivalent provided that solutions are quite smooth. Assume that \( u, \rho_0, \sigma \) is a smooth solution of (43)-(48). From (43) it follows

\[
\int_{\Omega^c} (-\text{div}\sigma(u) - f)(v - u) = 0 \ \forall v \in K_0.
\]

Hence,

\[
\int_{\Omega^c} \sigma(u) \varepsilon(v - u) - \int_{\Omega^c} f(v - u) + \int_{\Gamma^c} [\sigma n \cdot (v - u)] + \int_{\gamma} [\sigma \nu](v - u) = 0. \tag{52}
\]
Since from (47), (48) we derive
\[ \int_{\Gamma} [\sigma n \cdot (v - u)] + \int_{\gamma} [\sigma v](v - u) \leq -\int_{\omega} g(\rho - \rho_0) \quad \forall (v, \rho) \in K_0, \]
the relation (52) implies the variational inequality (51) as needed. In so doing we change the domain of integration \( \Omega_\gamma^\varepsilon \) by \( \Omega^\varepsilon \). This is possible since \([u] = 0\) on \( \gamma \).

Now we check that (43)-(48) can be derived from (50)-(51). Substitution of \((v, \rho) = (u \pm \varphi, \rho_0), \varphi \in C_0^\infty(\Omega_\gamma^\varepsilon)^2\), in (51) as test functions implies the equilibrium equations (43) holding in the distributional sense. Substituting next \((v, \rho) = (0, 0), (v, \rho) = 2(u, \rho_0)\) in (51) as test functions we derive
\[ \int_{\Omega_\gamma^\varepsilon} \sigma(u) \varepsilon(u) - \int_{\Omega_\gamma^\varepsilon} f u - \int_{\omega} g \rho_0 = 0. \quad (53) \]

We change the domain of integration \( \Omega^\varepsilon \) by \( \Omega_\gamma^\varepsilon \). Integration by parts in (53) yields (47). By (53), from (51) we have
\[ \int_{\Omega_\gamma^\varepsilon} \sigma(u) \varepsilon(v) - \int_{\Omega_\gamma^\varepsilon} f v - \int_{\omega} g \rho \geq 0 \quad \forall (v, \rho) \in K_0. \]
From this inequality, the relation (48) follows after the integration by parts. Thus (43)-(48) can be obtained from (50)-(51).

In order to find the derivative of the energy functional with respect to the crack length, we should consider the problems perturbed to (43)-(48). Let \( \lambda \) be a small parameter characterizing the crack length change, and \( \Gamma_\varepsilon^\lambda = \{0\} \times (0, 1 + \lambda) \) be a perturbed crack. For any fixed \( \lambda \) it is possible to prove the existence of solutions to the perturbed (with respect to (43)-(48)) problem. To this end, denote \( \Omega_{\varepsilon, \gamma}^\lambda = \Omega \setminus (\bar{\Gamma}_\varepsilon^\lambda \cup \bar{\gamma}), \Omega_{\varepsilon, \gamma}^\lambda = \Omega \setminus \bar{\Gamma}_\varepsilon^\lambda \). We have to find functions \( u^\lambda = (u_1^\lambda, u_2^\lambda), \rho_0^\lambda \in R(\omega), \sigma^\lambda = \{\sigma_{ij}^\lambda\}, i, j = 1, 2, \)
such that
\[-\text{div}\sigma^\lambda = f \quad \text{in} \quad \Omega_{\gamma}^\lambda, \quad (54)\]
\[\sigma^\lambda - A\varepsilon(u^\lambda) = 0 \quad \text{in} \quad \Omega^\lambda, \quad (55)\]
\[u^\lambda = 0 \quad \text{on} \quad \Gamma; \quad u^\lambda = \rho_0^\lambda, \quad \text{on} \quad \gamma, \quad (56)\]
\[[u^\lambda]n \geq 0 \quad \text{on} \quad \Gamma_{c}\lambda, \quad (57)\]
\[
\int_{\Gamma_{c}^\lambda} [\sigma^\lambda n \cdot u] + \int_{\gamma} [\sigma^\lambda \nu] \rho_0^\lambda + \int_{\omega} g \rho_0^\lambda = 0, \quad (58)
\]
\[
\int_{\Gamma_{c}^\lambda} [\sigma^\lambda n \cdot \bar{u}] + \int_{\gamma} [\sigma^\lambda \nu] \bar{\rho} + \int_{\omega} g \bar{\rho} \leq 0 \quad \forall (\bar{u}, \bar{\rho}) \in K_0^\lambda, \quad (59)
\]
where the set of admissible displacements $K_0^\lambda$ is defined as follows
\[K_0^\lambda = \{(v, \rho) \in H_0^\lambda \mid [v]n \geq 0 \text{ on } \Gamma_{c}^\lambda; \quad v|_{\gamma} = \rho, \quad \rho \in R(\omega)\}, \quad (60)\]

Like in problem (43)-(48) it is possible to prove the existence of solution to (54)-(59) and, therefore, to define the energy functional
\[E(\Omega^\lambda; g) = \frac{1}{2} \int_{\Omega^\lambda} \sigma(u^\lambda)\varepsilon(u^\lambda) - \int_{\Omega^\lambda} fu^\lambda - \int_{\omega} g \rho_0^\lambda. \quad (61)\]

Moreover, we can apply the technique developed in [3], [4], [12], [18] for finding the derivative of the energy functional with respect to $\lambda$. Namely, the following formula holds true
\[
\frac{d}{d\lambda} E(\Omega^\lambda; g)|_{\lambda=0} = \int_{\Omega^c} \left\{ \frac{1}{2} \varepsilon_{kl}(u)\varepsilon_{ij}(u)(a_{ijkl}\theta)_{,2} - \sigma_{ij}(u)u_{i,2}\theta_{,j} \right\} - \int_{\Omega^c} (\theta f_i)_{,2}u_i, \quad (61)
\]
where $\theta$ is an arbitrarily smooth function such that $\theta = 1$ in a neighborhood of the point $(0, 1)$, and $\theta = 0$ outside of a neighborhood of the point $(0, 1)$. Note that there is no dependence of the derivative (61) on the choice of $\theta$ with the prescribed properties. Define the cost functional

$$J(g) = \frac{d}{d\lambda} E(\Omega^{\lambda}; g)|_{\lambda=0},$$

where $g \in G$, and $G \subset L^2(\omega)^2$ is a bounded and weakly closed set. We should remark that $J(g) \leq 0 \forall g \in G$. Moreover, $J(g) = 0$ for a particular case $f = 0$, $g = 0$ (if $0 \in G$) since the solution of the problem (43)-(48) is trivial in this case.

Now we consider the optimal control problem

$$\sup_{g \in G} J(g). \quad (62)$$

Thus the problem (62) consists in finding the most safe loading $g \in G$ from the standpoint of the Griffith criterion. Note that also minimizing the cost functional is reasonable. The latter corresponds to maximizing the energy release rate, a choice which is suitable in applications where crashworthiness is at issue.

**Theorem 5.1** There exists a solution of the optimal control problem (62).

**Proof** Let $g^n \in G$ be a maximizing sequence. Since $G$ is a bounded and weakly closed set we can assume that, as $n \to \infty$,

$$g^n \to g_0 \text{ weakly in } L^2(\omega)^2, \ g_0 \in G. \quad (63)$$

For any $n$ there exists a solution of the problem

$$(u^n, \rho^n_0) \in K_0, \quad (64)$$

$$\int_{\Omega^c} \sigma(u^n)\varepsilon(v - u^n) - \int_{\Omega^c} f(v - u^n) - \int_{\omega} g^n(\rho - \rho^n_0) \geq 0 \ \forall (v, \rho) \in K_0. \quad (65)$$
By (65), the following equality holds

$$\int_{\Omega^c} \sigma(u^n)\varepsilon(u^n) - \int_{\Omega^c} f u^n - \int_{\omega} g^n \rho^n_0 = 0. \quad (66)$$

Since $u^n = \rho^n_0$ on $\gamma$, and

$$\|\rho^n_0\|_{H^1(\omega)}^2 = \left( \int_{\gamma} (\rho^n_0)^2 \right)^{1/2},$$

by the imbedding theorem, we have

$$\|\rho^n_0\|_{H^1(\omega)}^2 = \|\rho^n_0\|_{L^2(\gamma)}^2 = \|u^n\|_{L^2(\gamma)}^2 \leq c\|u^n\|_{H^1_\Gamma(\Omega^c)^2}. \quad (67)$$

Consequently, from (66) we derive, uniformly in $n$,

$$\|u^n\|_{H^1_\Gamma(\Omega^c)^2} \leq c. \quad (68)$$

Due to (67), (68), choosing a subsequence with the previous notation, we assume that as $n \to \infty$

$$u^n \to u \text{ weakly in } H^1_\Gamma(\Omega^c)^2, \quad (69)$$

$$\rho^n_0 \to \rho_0 \text{ strongly in } L^2(\omega)^2, \quad (70)$$

Note that we use here a compact imbedding of $H^1(\omega)^2$ in $L^2(\omega)^2$. Consequently, by (63), (69), (70), a passage to the limit as $n \to \infty$ can be performed in (64), (65) which provides

$$(u, \rho_0) \in K_0, \quad (71)$$

$$\int_{\Omega^c} \sigma(u)\varepsilon(v-u) - \int_{\Omega^c} f(v-u) - \int_{\omega} g_0(\rho-\rho_0) \geq 0 \quad \forall (v, \rho) \in K_0. \quad (72)$$

Thus, we conclude $u = u(g_0), \rho_0 = \rho_0(g_0)$. In addition to (69), a strong convergence of $u^n$ in $H^1_\Gamma(\Omega^c)^2$ takes place. Indeed, take
\((v, \rho) = (u, \rho_0)\) in (65), next take \((v, \rho) = (u^n, \rho^n_0)\) in (72) as test functions. These substitutions imply

\[
\int_{\Omega^c} \sigma(u^n) \varepsilon(u - u^n) - \int_{\Omega} f(u - u^n) - \int_{\omega} g^n(\rho_0 - \rho^n_0) \geq 0,
\]

\[
\int_{\Omega^c} \sigma(u) \varepsilon(u^n - u) - \int_{\Omega} f(u^n - u) - \int_{\omega} g_0 \rho^n_0 - \rho_0 \geq 0,
\]

thus

\[
\int_{\Omega^c} \sigma(u^n - u) \varepsilon(u^n - u) - \int_{\omega} (g_0 - g^n)(\rho_0 - \rho^n_0) \leq 0. \tag{73}
\]

The inequality (73) implies

\[
\|u^n - u\|_{H_1^1(\Omega^c)^2} \leq c\|g_0 - g^n\|_{L^2(\omega)^2}\|\rho_0 - \rho^n_0\|_{L^2(\gamma)^2}. \tag{74}
\]

Due to imbedding theorems, we can assume that \(u^n \to u\) strongly in \(L^2(\gamma)^2\). Consequently, by the relation

\[
\frac{1}{2} \varepsilon_{kl}(u^n)\varepsilon_{ij}(u^n)(a_{ijkl}\theta)_{,2} - \sigma_{ij}(u^n)u^n_{i,2}\theta_{,j} - \int_{\Omega^c}(\theta f_i)_{,2}u^n_i,
\]

and, due to (75), we obtain

\[
J(g^n) \to J(g_0).
\]

Consequently, the limiting function \(g_0 \in G\) solves the optimal control problem (62). Theorem 5.1 is proved.
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