Solving Nonlinear Multiobjective Bilevel Optimization Problems with Coupled Upper Level Constraints

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Abstract

For bilevel optimization problems with a multiobjective optimization problem on each level we consider a very general formulation with the upper-level constraint being dependent on the lower-level variable. Extending results of an earlier work we give connections between the solution sets of several multiobjective optimization problems and the induced set of the multiobjective bilevel problem. Using these results together with a refinement strategy involving new sensitivity results for the parameter-dependent \(\varepsilon\)-constraint scalarization we develop an algorithm for solving such problems. We also demonstrate the applicability of our algorithm on two test problems.

1 Introduction

Bilevel optimization has become an important field of mathematical programming (see the monographs by Dempe [7] and Bard [2] as well as [6, 9], and the bibliography reviews by Calamai and Vicente [35] and Dempe [8]).

Applications for bilevel optimization problems are various, see the survey in [6] or [26]. For instance in toll-setting ([6, 21]) the hierarchical structure of bilevel optimization is appropriate for modeling this problem: the aim is to maximize the revenue raised from tolls set on some links of a transportation network. The network managers decide about the toll levels while this influences of course the travel cost. The travelers react and minimize their total travel cost consisting of standard costs as time and distance and further of the tolls. Thus the toll levels should not be chosen

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too high because otherwise the travelers are deterred from using the toll links while high revenues are still the aim. This demonstrates the two-way influence typically for bilevel optimization problems.

Lately the natural extension of (scalar-valued) bilevel optimization problems to bilevel problems with vector-valued objective functions on each level has been considered. Applications for multiobjective bilevel optimization problems are given e.g. in [36] and [15]. However very few papers are related to solution methods for multiobjective bilevel optimization problems: in [27] Nishizaki and Sakawa present a method for linear multiobjective bilevel optimization problems, Shi and Xia, [31, 32], give a procedure for nonlinear bilevel problems based on interaction with the decision maker, while Osman, Abo-Sinna et al. ([1, 28]) use fuzzy set theory for convex problems. Teng et al., [34], discuss a convex multiperson multiobjective bilevel problem and in [22] Liou et al. examine existence results and equivalences between vector bilevel optimization problems and multiobjective optimization problems with equilibrium constraints.

In [15, 19] nonlinear vector-valued bilevel optimization problems are discussed and solved, there with the upper-level constraint being uncoupled from the lower-level variable. Uncoupled upper-level constraints are widely considered see for instance [2, p.303],[7, pp.123f.],[8, 10, 12]. However, a more general formulation of a bilevel optimization problem is reached by allowing the constraint of the upper level to depend on the lower-level variable, compare [2, 7, 9, 35].

In this paper we generalize the results of [15] for those types of problems called general bilevel problems as in [6]. Thereby, a theoretical gap arises, but using a modified formulation we can close this gap. We can show equality between the induced set of the bilevel problem and the intersection of the minimal solution set of a multiobjective optimization problem and the upper-level constraint set. An algorithm is developed for solving generalized multiobjective bilevel optimization problems especially for the case of a one-dimensional upper-level variable and a biobjective optimization problem on the lower level. Numerical results for two test problems are presented. Note, that all results are gained without convexity assumptions and without Karush-Kuhn-Tucker type reformulations of the lower level.

The paper is organized as follows. In Section 2 we introduce the notations of bilevel and multiobjective optimization and we describe the discussed problem. Section 3 includes the main theoretical results and in Section 4 an algorithm for solving the general multiobjective bilevel optimization problem is given. In Section 5 we illustrate the technique with some numerical examples and finally we report our conclusions.

2 Basic concepts and general formulation

Bilevel optimization problems have a hierarchical structure with an optimization problem on the upper level and another on the lower level. Due to that hierarchical structure multilevel problems are closely related to Stackelberg games in game theory ([33]). Yet there, the lower-level problem is an equilibrium problem (see e.g. [6]) while in bilevel optimization we have an optimization problem on the lower level. On that
lower level an objective function \( f : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{m_1} \) is minimized w. r. t. \( x \in \mathbb{R}^{n_1} \) subject to the constraint \((x, y) \in G\) with \( G \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \) a non-empty set:

\[
\min_{x} f(x, y)
\]
subject to the constraint
\[
(x, y) \in G.
\]

Here, \( y \in \mathbb{R}^{n_2} \) is a parameter of that lower-level optimization problem. The variable \( x \) (resp. the variable \( y \)) is called lower-level (resp. upper-level) variable. Let the solution set of the lower-level problem (also called lower-level reaction set) be denoted as

\[
\Psi(y) := \arg\min_{x} \{ f(x, y) \mid (x, y) \in G \}.
\]

On this set \( \Psi(y) \) the upper-level optimization problem with the objective function \( F : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{m_2} \) \((m_2 \in \mathbb{N})\) is defined:

\[
\min_{x, y} F(x(y), y)
\]
subject to the constraints
\[
\begin{align*}
  x(y) &\in \Psi(y), \\
  (x, y) &\in \tilde{G}
\end{align*}
\]

with \( \tilde{G} \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \) a compact set. This problem formulation of a (general) bilevel optimization problem is based on the so-called optimistic approach as we minimize on the upper level w. r. t. \( x \) and \( y \) (as widely used for instance in [4, 21, 22]) and not only w. r. t. \( y \). This approach is often used if it cannot be guaranteed that the lower-level minimal solution is unique. For more details see e. g. [7],[9, p.5],[10, p.506],[27, p.166]. The constraint set \( \Omega \) of the upper-level problem, also called induced set, is given by

\[
\Omega = \{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \mid x \in \Psi(y), \ (x, y) \in \tilde{G}\}.
\]

In problem (1) the upper-level constraint \((x, y) \in \tilde{G}\) involves variables from both levels in contrast to the less general formulation used in [15] with the constraint \( y \in \tilde{G} \) independent on \( x \). Notice, that in the first publication which used the term bilevel optimization ([5]) the problems did not involve joint upper-level constraints (see also [6]). Some papers on bilevel optimization do not assume constraints on the upper level at all, see e. g. [4, 22].

Further notice that the constraint \((x, y) \in \tilde{G}\) from the upper level (also if it is just a constraint dependent on \( x \) as \( x \in \tilde{G} \)) cannot be just moved to the lower level as the following example demonstrates.

**Example 1** We consider the bilevel optimization problem

\[
\min_{x, y} y + \frac{1}{2} x
\]
subject to the constraints
\[
\begin{align*}
  x &\geq 2 - y, \\
  y &\in [0, 2], \\
  x &\in \arg\min_{x} \{ x \in \mathbb{R} \mid x \in [0, 2], \ x \geq y \}.
\end{align*}
\]
The lower-level minimal solution depended on \( y \in [0, 2] \) is thus \( x(y) = y \) and hence only the points \((x(y), y) = (y, y)\) for \( y \in [1, 2] \) are feasible for the upper level. This leads to the optimal value \(3/2\) of the upper-level function and the minimal solution \((x, y) = (1, 1)\).

Moving the constraint \( x \geq 2 - y \) instead to the lower level, i.e. considering the problem

\[
\min_x \{ x \mid x \in [0, 2], \, x \geq y, \, x \geq 2 - y \}
\]
on the lower level leads to the lower-level minimal solutions

\[
x(y) = \begin{cases} 
2 - y, & \text{for } y \in [0, 1], \\
y, & \text{for } y \in ]1, 2].
\end{cases}
\]

Then the upper-level optimal value is \(1\) with minimal solution \((x, y) = (2, 0)\).

This has of course also a practical interpretation. The same constraint on the lower level restricting the feasible set on the lower level has a different meaning as on the upper level. There, the feasibility is restricted after the determination of the minimal solution of the lower level and is thus an implicit constraint for the upper level problem. For more details see [7, pp.25f].

As in [15] we consider vector-valued objective functions on each level, i.e. let \(m_1, m_2 \geq 2\). Bonnel and Morgan, [3], denote a bilevel optimization problem with a scalar-valued optimization problem on the upper level and a vector-valued optimization problem on the lower level a semivectorial bilevel optimization problem.

In the following we give a short introduction to multiobjective optimization. For more details see e.g. the books by Jahn, [17], or Sawaragi et al., [30]. For defining minimality of a multiobjective optimization problem a partial ordering in the image space has to be introduced. Let \(K \subset \mathbb{R}^m\) be a convex cone, i.e. \(\lambda(x + y) \in K\) for all \(\lambda \geq 0\), \(x, y \in K\). Then a partial ordering in \(\mathbb{R}^m\) is defined by

\[
x \leq_K y \iff y - x \in K.
\]

The partial ordering is antisymmetric if the cone \(K\) is pointed, i.e. if \(K \cap (-K) = \{0_m\}\). For example the natural ordering is defined by the pointed convex cone \(\mathbb{R}^m_+.\) In the following let \(K\) be a closed pointed convex cone. We consider the multiobjective optimization problem

\[
\min_x f(x) \\
\text{subject to the constraint} \\
x \in \Omega
\]

with a constraint set \(\Omega \subset \mathbb{R}^n\) and the objective function \(f : \mathbb{R}^n \to \mathbb{R}^m, \, n, m \in \mathbb{N}, \, m \geq 2\).

A point \(\bar{x} \in \Omega\) is called \(K\)-minimal for (3) if

\[
(f(\bar{x}) - K) \cap f(\Omega) = \{f(\bar{x})\}.
\]

For an illustration (for \(m = 2\)) see Fig. 1. For \(\text{int}(K) \neq \emptyset\) a point \(\bar{x} \in \Omega\) is called
weakly $K$-minimal for (3) if
\[(f(\bar{x}) - \text{int}(K)) \cap f(\Omega) = \emptyset.\]

For the case of an empty interior of the cone $K$ weak minimality can also be defined w.r.t. the intrinsic core, see [17, p. 109], [29].

We denote the set of all $K$-minimal points as $\mathcal{M}(f(\Omega), K)$ and the set of all weakly $K$-minimal points as $\mathcal{M}_w(f(\Omega), K)$. The set $\mathcal{E}(f(\Omega), K) := \{f(x) \in \mathbb{R}^m \mid x \in \mathcal{M}(f(\Omega), K)\}$ is called efficient set and the set $\mathcal{E}_w(f(\Omega), K) := \{f(x) \in \mathbb{R}^m \mid x \in \mathcal{M}_w(f(\Omega), K)\}$ weakly efficient set. For $K = \mathbb{R}^m_+$ the $K$-minimal points are denoted as Edgeworth-Pareto (EP)-minimal points, too.

For the considered general multiobjective bilevel optimization problem (1) we assume that the partial ordering on the upper level is given by the closed pointed convex cone $K^2 \subset \mathbb{R}^{m_2}$ and on the lower level by the closed pointed convex cone $K^1 \subset \mathbb{R}^{m_1}$.

### 3 Theoretical results

In this section we discuss the relationship between the induced set $\Omega$ as defined in (2) and the solution set of a multiobjective optimization problem. First, we generalize the approach used in [15] and thus we consider the multiobjective optimization problem

\[
\min_{x,y} \hat{f}(x, y) := \begin{pmatrix} f(x, y) \\ y \end{pmatrix}
\]

subject to the constraints

\[(x, y) \in G, \quad (x, y) \in \hat{G} \tag{4}\]

w.r.t. the ordering cone $\hat{K} := K^1 \times \{0_{n_2}\} \subset \mathbb{R}^{m_1} \times \mathbb{R}^{n_2}$. We have just added the upper level variable to the lower level objective functions and we have collected all constraints. Thus this is a multiobjective optimization problem with $m_1 + n_2$ objectives. Let

\[
\hat{G} := \{(x, y) \in \mathbb{R}^{m_1} \times \mathbb{R}^{n_2} \mid (x, y) \in G, \ (x, y) \in \hat{G}\} = G \cap \hat{G}
\]

denote the constraint set of problem (4). Then the $\hat{K}$-minimal solution set of (4) is denoted as

\[
\hat{\mathcal{M}} := \mathcal{M}(\hat{f}(\hat{G}), \hat{K}).
\]
This set $\hat{M}$ has a close connection to the induced set $\Omega$ as the following theorem shows. However we get no equivalence result like in the uncoupled case in [15] Thm. 3.1

**Theorem 3.1** Let $\hat{M}$ be the set of $\hat{K}$-minimal points of the multiobjective optimization problem (4) with $\hat{K} = K^1 \times \{0_m\}$ and let $\Omega$ be the induced set of the multiobjective bilevel optimization problem (1). Then it is $\Omega \subset \hat{M}$.

**Proof.** We have

$(\bar{x}, \bar{y}) \in \Omega$ $\iff$ $(\bar{x}, \bar{y}) \in \hat{G} \land \bar{x} \in \Psi(\bar{y})$

$\iff (\bar{x}, \bar{y}) \in G \land (\bar{x}, \bar{y}) \in \hat{G} \land (\bar{y}, \bar{y}) \in G$ with $f(\bar{x}, \bar{y}) \in f(x, y) + K^1 \setminus \{0_m\}$

$\Rightarrow (\bar{x}, \bar{y}) \in \hat{G} \land (\bar{x}, \bar{y}) \in G$ with $(x, y) \in \hat{G}$ and $f(\bar{x}, \bar{y}) \in f(x, y) + K^1 \{0_m\}$

$\iff (\bar{x}, \bar{y}) \in \hat{G} \land (\bar{x}, \bar{y}) \in G$ with $y = \bar{y}$ and $f(\bar{x}, \bar{y}) \in f(x, y) + K^1 \{0_m\}$

$\iff (\bar{x}, \bar{y}) \in \hat{G} \land (\bar{x}, \bar{y}) \in G$ with $f(\bar{x}, \bar{y}) \in f(x, y) + \hat{K} \{0_m\}$

$\iff (\bar{x}, \bar{y}) \in \hat{M}$

$\square$

In [15, Thm. 4.1] it is shown that for an uncoupled upper level constraint, i.e. having the constraint $y \in \hat{G} \subset \mathbb{R}^{n_2}$ instead of $(x, y) \in \hat{G} \subset \mathbb{R}^{n_1+n_2}$ on the upper level, equality holds, i.e. $\Omega = \hat{M}$. However, generally we have a strict inclusion as the following example demonstrates.

**Example 2** We consider the biobjective bilevel optimization problem

$$\min_{x, y} F(x, y) = \begin{pmatrix} x_1 & y \\ x_2 & \end{pmatrix}$$

subject to the constraints

$x = x(y) \in \arg\min_x \left\{ f(x, y) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \big| (x, y) \in G \right\}$,

$(x, y) \in \hat{G}$

with $n_1 = 2$, $n_2 = 1$, $K^1 = K^2 = \mathbb{R}^2_+$, $m_1 = m_2 = 2$,

$G := \{(x, y) \in \mathbb{R}^3 \mid \|x\|_2^2 \leq y^2\}$

and

$\hat{G} := \{(x, y) \in \mathbb{R}^3 \mid 0 \leq y \leq 1, \ x_1 + x_2 \geq -1\}$.

Then it is

$\Psi(y) = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid \|x\|_2^2 = y^2, \ x_1 \leq 0, \ x_2 \leq 0\}$.
and thus the induced set is
\[ \Omega = \{(x, y) \in \mathbb{R}^3 \mid \|x\|_2^2 = y^2, \ x_1 \leq 0, \ x_2 \leq 0, \ 0 \leq y \leq 1, \ x_1 + x_2 \geq -1\}. \]

Let \( \hat{\mathcal{M}} \) be the set of \((\mathbb{R}_+^2 \times \{0\})\)-minimal points of the tricriteria optimization problem
\[
\min_{x,y} \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix}
\]
subject to the constraints
\[
(x, y) \in G \cap \hat{G} = \{(x, y) \in \mathbb{R}^3 \mid 0 \leq y \leq 1, \ x_1 + x_2 \geq -1, \ \|x\|_2 \leq y^2\},
\]
as in (4).

Then, for \((\bar{x}, \bar{y}) = (-\frac{1}{2}, -\frac{1}{2}, 1)^T\), it is \((\bar{x}, \bar{y}) \in \hat{\mathcal{M}}\) because \((\bar{x}, \bar{y}) \in G \cap \hat{G}\) and there is no \((x, y) \in G \cap \hat{G}\) with
\[
\hat{f}(\bar{x}, \bar{y}) < \hat{f}(x, y) + \hat{K} \setminus \{0_3\}
\]
\[
\Leftrightarrow \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{y} \end{pmatrix} \in \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} + (\mathbb{R}_+^2 \times \{0\}) \setminus \{0_3\}
\]
\[
\Leftrightarrow \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} \in \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \mathbb{R}_+^2 \setminus \{0_2\} \land y = \bar{y}.
\]

The set \(\{x \in \mathbb{R}^2 \mid (x, y) \in G \cap \hat{G}, \ y = 1\}\) is drawn in Fig. 2.

However it is \((\bar{x}, \bar{y}) \notin \Omega\) because \(\|\bar{x}\|_2^2 = \frac{1}{2} \neq 1 = \bar{y}^2\).

In this example, as the induced set can be determined explicitly, the minimal solution set can be calculated by solving the biobjective optimization problem \(\min_{x,y} \{F(x, y) \mid (x, y) \in \Omega\}\). We get as solution set of the multiobjective bilevel optimization problem:
\[
S^{\min} = \left\{(x_1, x_2, y) \mid x_1 = -1 - x_2, \ x_2 = -\frac{1}{2} \pm \frac{1}{4}\sqrt{8y^2 - 4}, \ y \in \left[\frac{\sqrt{2}}{2}, 1\right] \right\}.
\]
Figure 3: Solution set of the biobjective bilevel optimization problem of Example 2 and its image under the objective functions of the upper level.

The image \( \{ F(x_1, x_2, y) \mid (x_1, x_2, y) \in S^{\text{min}} \} \) of this set under the upper-level objective functions is equal to

\[
\left\{ (z_1, z_2) \in \mathbb{R}^2 \mid z_1 = -1 - z_2 - y, \ z_2 = -\frac{1}{2} \pm \frac{1}{4} \sqrt{8y^2 - 4}, \ y \in \left[ \frac{\sqrt{2}}{2}, 1 \right] \right\}.
\]

These sets are shown in Fig. 3.

Problem (4) has the same structure if we formulate it to the bilevel optimization problem

\[
\begin{aligned}
\min_{x,y} & \quad F(x, y) \\
\text{subject to the constraints} & \quad x \in \arg\min_x \{ f(x, y) \mid (x, y) \in G \cap \tilde{G} \}.
\end{aligned}
\]

As here no upper level constraint exists we have according to Theorem 4.1 in [15] that the \( \hat{K} \)-minimal solution set \( \hat{M} \) of (4) equals the induced set \( \Omega \) of (5). Hence, under the conditions of Theorem 3.1 we cannot gain equality in general as otherwise this would imply that the induced set \( \Omega \) of (1) and (5) are equal. This would mean that it makes no difference on which level the constraint \( (x, y) \in \tilde{G} \) is given in contradiction to the Example 1.

Of course it is \( \hat{M} \subset G \cap \tilde{G} \) and thus \( \hat{M} \subset \tilde{G} \). Therefore \( \hat{M} \cap \tilde{G} = \hat{M} \) and we also have only \( \Omega \subset \hat{M} \cap \tilde{G} \).

Theorem 3.1 does not produce an equivalent formulation for the induced set \( \Omega \). However this can be reached by considering the following modified multiobjective optimization problem instead of problem (4):

\[
\begin{aligned}
\min_{x,y} & \quad \hat{f}(x, y) = \left( \begin{array}{c} f(x, y) \\ y \end{array} \right) \\
\text{subject to the constraints} & \quad (x, y) \in G, \\
& \quad y \in \tilde{G}^y
\end{aligned}
\]

w.r.t. the ordering cone \( \hat{K} \). Here let the compact set \( \tilde{G}^y \subset \mathbb{R}^{n_2} \) be an arbitrary compact set such that

\[
\{ y \in \mathbb{R}^{n_2} \mid \exists x \in \mathbb{R}^{n_1} \text{ such that } (x, y) \in \tilde{G} \} \subset \tilde{G}^y.
\]
Notice, that we have assumed the set $\tilde{G}$ to be compact and thus $\tilde{G}^y$ exists. Let
\[ \tilde{G}^0 := \{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \mid (x, y) \in G, \ y \in \tilde{G}^y \} \supset \tilde{G} \]
denote the constraint set of (6) and let
\[ \hat{M}^0 := M(\hat{f}(\tilde{G}^0), \hat{K}) \]
be the $\hat{K}$-minimal solution set of (6). Then we have the equality
\[ \hat{M}^0 \cap \tilde{G} = \Omega, \]
as the following theorem shows.

**Theorem 3.2** Let $\hat{M}^0$ be the set of $\hat{K}$-minimal points of the multiobjective optimization problem (6) with $\hat{K} = K^1 \times \{0_{n_2}\}$ and let $\Omega$ be the induced set of the multiobjective bilevel optimization problem (1) with the upper-level constraint $(x, y) \in \tilde{G}$. Then it is $\Omega = \hat{M}^0 \cap \tilde{G}$.

**Proof.**

We have
\[ (\bar{x}, \bar{y}) \in \Omega \iff (\bar{x}, \bar{y}) \in \tilde{G} \land (\bar{x}, \bar{y}) \in G \land (\bar{f}(\bar{x}, \bar{y}) \in G \land f(\bar{x}, \bar{y}) + K^1 \setminus \{0_{m_1}\}) \]
\[ (\bar{x}, \bar{y}) \in G \land (\bar{x}, \bar{y}) \in \tilde{G}^y \iff (\bar{x}, \bar{y}) \in \tilde{G} \land (\bar{f}(\bar{x}, \bar{y}) + K^1 \setminus \{0_{m_1}\}) \]
\[ (\bar{x}, \bar{y}) \in \tilde{G} \land (\bar{x}, \bar{y}) \in G \land (\bar{f}(\bar{x}, \bar{y}) + K^1 \setminus \{0_{m_1}\}) \]
\[ (\bar{x}, \bar{y}) \in \tilde{G} \land (\bar{x}, \bar{y}) \in G \land (\bar{f}(\bar{x}, \bar{y}) + K^1 \setminus \{0_{m_1}\}) \]
\[ (\bar{x}, \bar{y}) \in \tilde{G} \land (\bar{x}, \bar{y}) \in G \land (\bar{f}(\bar{x}, \bar{y}) + K^1 \setminus \{0_{m_1+n_2}\}) \]
\[ (\bar{x}, \bar{y}) \in \tilde{G} \land (\bar{x}, \bar{y}) \in G \land (\bar{f}(\bar{x}, \bar{y}) + K^1 \setminus \{0_{m_1+n_2}\}) \]
\[ (\bar{x}, \bar{y}) \in \tilde{G} \land (\bar{x}, \bar{y}) \in G \land (\bar{f}(\bar{x}, \bar{y}) + K^1 \setminus \{0_{m_1+n_2}\}) \]
\[ (\bar{x}, \bar{y}) \in \tilde{G} \land (\bar{x}, \bar{y}) \in G \land (\bar{f}(\bar{x}, \bar{y}) + K^1 \setminus \{0_{m_1+n_2}\}) \]
\[ \square \]

Notice that we get the result of Theorem 3.2 also if we discard the constraint $y \in \tilde{G}^y$ in (6). However then the solution set of (6) may contain points $(\bar{x}, \bar{y})$ with $\bar{y} \notin \tilde{G}^y$. These points are of interest as $y \notin \tilde{G}^y$ implies $(\bar{x}, \bar{y}) \notin \tilde{G}$ and hence these points are no element of $\hat{M}^0 \cap \tilde{G}$. Thus, considering (6) without the constraint $y \in \tilde{G}^y$ may mean that the set $\hat{M}^0$ is unnecessary large.

**4 Numerical method and sensitivity results**

The basic structure which we use for solving the problem (1) is the following: first we determine an approximation of the induced set $\Omega$. Then, this approximation is mapped under the upper-level objective $F$ and the $K^2$-minimal points are determined. These points of the set $\Omega$ are, so far, the best approximations of the minimal solutions of the bilevel optimization problem. Then, in a second step, we refine the discretization of the induced set $\Omega$ around the $K^2$-minimal points. Thereby we use the result of Theorem 3.2.
4.1 Approximation of the induced set

In the first step we determine an approximation of the induced set $\Omega$. This approximation should cover the whole set with a given accuracy. For convenience we recall $\Omega$:

$$\Omega = \{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} | x \in \Psi(y), (x, y) \in \tilde{G}\}.$$  

We propose two possibilities for determining an approximation of $\Omega$. First, we can use the results of Theorem 3.2 and determine an approximation of the solution set of the multiobjective optimization problem (6). For this approach we need an efficient method for solving multiobjective optimization problems providing an approximation with a high quality. A good quality, i.e. almost equidistant approximation points, is needed, as the minimal solutions are also (partly) points of the induced set $\Omega$, which should be covered well. Afterwards only those minimal solutions $(x, y)$ are of interest for which it holds $(x, y) \in \tilde{G}$. Using this idea demands thus the high quality solution of an optimization problem with $m_1 + n_2$ objectives.

As this is a hard task already for two objectives we use in the algorithm presented in Section 4.3 another approach. We determine the set $\Psi(y)$ for $y \in \tilde{G}^y$. Then, we can easily approximate $\Omega$ due to

$$\Omega = \{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} | x \in \Psi(y), (x, y) \in \tilde{G}\} \subset \{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} | x \in \Psi(y), y \in \tilde{G}^y\}.$$  

Of course we cannot determine $\Psi(y)$ (which means the determination or at least approximation of the $K^1$-minimal solution set of the lower-level multiobjective optimization problem) for infinitely many parameters $y$. However we can do this for a selection of parameters $\{y_1, \ldots, y_k\} \subset \tilde{G}^y$, e.g. equidistantly chosen. As the number of the lower-level problems which have to be solved increases rapidly with the dimension of $\tilde{G}^y$ (and with the discretization fineness), we consider here only one-dimensional upper-level variables $y$, i.e. we assume $n_2 = 1$.

For each selected parameter $y^i \in \tilde{G}^y$ the $K^1$-minimal solution set of the lower-level problem has to be approximated. For each $K^1$-minimal solution $x$ with $(x, y^i) \in \tilde{G}$ the point $(x, y^i)$ is an approximation point of the induced set $\Omega$. As we want a representative approximation of $\Omega$ the aim is again a nearly equidistant approximation of the $K^1$-minimal solution set of the multiobjective optimization problems on the lower level. For that, an efficient numerical method is needed. As this is, as already mentioned, very difficult task for a high number of objectives we assume that we have only two objectives on the lower level, i.e. $m_1 = 2$. In detail we assume the following:

**Assumption 1** Let $m_1 = 2$, $n_2 = 1$, and let $G$ be given by

$$G = \{(x, y) \in \mathbb{R}_+^{n_1} + 1 | g_j(x, y) \geq 0, j = 1, \ldots, p\}$$

($p \in \mathbb{N}$). Let the functions $f_1, f_2, g_j$, $j = 1, \ldots, p$, be twice continuously differentiable w. r. t. $(x, y)$. Let $\tilde{G}$ be compact such that $\tilde{G}^y = [c, d]$ as in (8) exists $(c, d \in \mathbb{R})$. Let $K^1 = \mathbb{R}_+^2$. 

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Under these assumptions a solution procedure is presented in [15] which generates almost equidistant approximations of the $K^1$-minimal solution set of the lower-level problem based on a scalarization approach. However any other procedure for solving multiobjective problems which generate high-quality approximations of the minimal solution set can be applied, too. Notice, that here we are interested in a high quality of the approximation of the minimal solution set and not of the image set of this set, the so-called efficient set. Generally in multiobjective optimization one is interested in a representative approximation of the efficient set as the decision maker elects solutions based on a comparison of the objective function values.

Let \( \{x^{i,1}, \ldots, x^{i,n^i}\}, n^i \in \mathbb{N} \), be such an approximation set of the lower level problem for the parameter \( y^i \). Then, if \((x^{i,j}, y^i) \in G (j = 1, \ldots, n^i)\) it is also \((x^{i,j}, y^i) \in \Omega\). After having solved the lower level problem for several parameters \( y \) we get hence a discrete approximation \( A \) of the induced set \( \Omega \) by

\[
A = \left( \bigcup_{i=1}^{k} \{ (x^{i,j}, y^i) \mid j = 1, \ldots, n^i \} \right) \cap \tilde{G}.
\]

Now, we calculate the set

\[
F(A) := \{ F(x, y) \mid (x, y) \in A \} \subset F(\Omega).
\]

Then we determine the non-dominated points of this set, i.e. we identify

\[
\mathcal{M}(F(A), K^2) = \{ (\bar{x}, \bar{y}) \in A \mid (F(\bar{x}, \bar{y}) - K^2) \cap F(A) = \{ F(\bar{x}, \bar{y}) \} \} \subset \Omega.
\]

This can be done using a Pareto-filter [24, 25] or the Jahn-Graef-Younes method, [18]. These points are approximations of the minimal solutions of the multiobjective bilevel optimization problem. For improving this approximation we refine the discretization of the induced set around these points in the next step.

### 4.2 Refinement of the induced set

In the previous section we have described how to determine a first approximation of the induced set \( \Omega \) and how to detect the interesting (the non-dominated) points of this set. Now, in a second step, we refine the approximation of the set \( \Omega \) around these points \( \mathcal{M}(F(A), K^2) \).

Let \((\bar{x}, \bar{y}) \in \mathcal{M}(F(A), K^2)\), i.e. \((\bar{x}, \bar{y}) \in \Omega = \hat{M}^0 \cap \tilde{G}\) according to Theorem 3.2. Thus the point \((\bar{x}, \bar{y})\) is a \(K\)-minimal solution of the multiobjective optimization problem (6), too. We are now interested in finding further points \((x, y) \in \Omega\) in a neighborhood of \((\bar{x}, \bar{y})\) such that

\[
\| (x, y) - (\bar{x}, \bar{y}) \| = \beta
\]

for a predefined distance \( \beta > 0 \) for doing a controlled refinement.

However instead of looking for further points \((x, y) \in \Omega\) we try to find further points \((x, y) \in \hat{M}^0\) with property (9). Then, afterwards, we select only those points \((x, y) \in \hat{M}^0\) with \((x, y) \in \tilde{G}\) (see Theorem 3.2).
For calculating points with property (9) we need a solution method for finding \( \hat{K} \)-minimal points of a multiobjective optimization problem which we can control in such a way that we can influence the position of the newly found points \((x, y)\).

Here, for that purpose, we use the well-known and wide spread \( \varepsilon \)-constraint scalarization \((P_k(\varepsilon))\), see [16, 23], to problem (6):

\[
\begin{align*}
\min_{x,y} & \ f_k(x, y) \\
\text{subject to the constraints} & \\
& \varepsilon_i - f_i(x, y) \geq 0 \quad \text{for } i \in \{1, \ldots, m_1\} \setminus \{k\} \\
& \varepsilon_i = f_i(x, y) \quad \text{for } i \in \{m_1 + 1, \ldots, m_1 + n_2\} \\
& (x, y) \in G \\
& y \in G_y \\
\end{align*}
\]

for an arbitrary \( k \in \{1, \ldots, m_1\} \). Then any minimal solution of \((P_k(\varepsilon))\) is weakly \( \hat{K} \)-minimal. Here weakly \( \hat{K} \)-minimal means minimal w. r. t. \((\text{int}(\mathbb{R}^m_+) \cup \{0_{m_1}\}) \times \{0_{n_2}\}\).

If the minimal solution is unique then it is even \( \hat{K} \)-minimal. Let \((\bar{t}, \bar{x})\) be a \( \hat{K} \)-minimal solution of (6) (for \( \hat{K} = \mathbb{R}^m_+ \times \{0_{n_2}\}\)), then \((\bar{t}, \bar{x})\) is also a minimal solution of \((P_k(\varepsilon))\) for instance for

\[
k = m_1, \quad \bar{\varepsilon}_i = \hat{f}_i(\bar{x}, \bar{y}), \quad i = 1, \ldots, m_1 + n_2, \quad i \neq m_1.
\]

If we know the connection between the parameter \( \varepsilon \) and the found minimal solution \((x(\varepsilon), y(\varepsilon))\) of problem \((P_k(\varepsilon))\) (which are also weakly \( \hat{K} \)-minimal solutions of (6)), we can control the position of the newly found points \((x, y) \in M^0\). The answer is delivered by the following sensitivity theorem which is an application of Thm. 3.2.2 and Cor. 3.2.4 in [11], but other sensitivity results as in [20] for the non-degenerated case, can be applied of course, too. Here we assume w. l. o. g. \( k = m_1 \), i. e. \( k = 2 \).

**Theorem 4.1** Let Assumption 1 hold. Let \((\bar{x}, \bar{y})\) be \( \hat{K} \)-minimal for (6) with \( \hat{K} = \mathbb{R}^m_+ \times \{0\}\) and \( \bar{y} \notin \{c, d\} \). Then \((\bar{x}, \bar{y})\) is a minimal solution of \((P_2(\varepsilon))\) with \( \bar{\varepsilon}_1 = \hat{f}_1(\bar{x}, \bar{y}), \quad \bar{\varepsilon}_3 = \bar{y} \). Let the second-order sufficient condition for a local minimum of \((P_2(\varepsilon))\) hold at \((\bar{x}, \bar{y})\) with Lagrange-multipliers \( \bar{\mu}_1 \in \mathbb{R}_+ \) to the constraint \( \bar{\varepsilon}_1 = \hat{f}_1(x, y) \geq 0, \quad \bar{\mu}_3 \in \mathbb{R} \) to \( \bar{f}_3(x, y) - \bar{\varepsilon}_3 = 0, \) and \( \mu \in \mathbb{R}^p_+ \) to \( g(x, y) \geq 0_p \). That is, let

\[
x^\top \nabla^2_x \mathcal{L}(\bar{x}, \bar{y}, \bar{\mu}_1, \bar{\nu}, \bar{\mu}_3, \bar{\varepsilon}) x > 0
\]

for all \( x \neq 0 \) with \( \nabla_x \hat{f}_1(\bar{x}, \bar{y})^\top x = 0 \) if \( \bar{\mu}_1 > 0 \), \( \nabla_x g_j(\bar{x}, \bar{y})^\top x = 0 \) if \( \bar{\nu}_j > 0 \), \( j = 1, \ldots, p \).

Here, \( \mathcal{L} \) denotes the Lagrange-function to \((P_2(\varepsilon))\) with

\[
\mathcal{L}(x, y, \mu_1, \nu, \mu_3, \varepsilon) = \hat{f}_2(x, y) - \mu_1(\varepsilon_1 - \hat{f}_1(x, y)) - \sum_{j=1}^{p} \nu_j g_j(x, y) + \mu_3(\bar{f}_3(x, y) - \varepsilon_3).
\]

Further let strict complementary slackness hold, i. e. let the Lagrange multipliers be strict positive for active inequalities. Let the gradients of the active constraints be linearly independent.
Then \((\bar{x}, \bar{y})\) is a local unique minimal solution of \((P_2(\bar{\varepsilon}))\) and there exists a \(\delta > 0\) such that the function \(\phi: N(\bar{\varepsilon}) \to B_3(\bar{x}, \bar{y}) \times B_3(\bar{\mu}_1, \bar{\nu}, \bar{\mu}_3)\) (for \(N(\bar{\varepsilon})\) a neighborhood of \(\bar{\varepsilon}\) and \(B_3(\cdot)\) a closed unit ball around the particular point with radius \(\delta\))

\[
\phi(\varepsilon) = (x(\varepsilon), y(\varepsilon), \mu_1(\varepsilon), \nu(\varepsilon), \mu_3(\varepsilon))
\]

has the following first order approximation

\[
\phi(\varepsilon) = \phi(\varepsilon) + \tilde{M}^{-1} \tilde{N} (\varepsilon - \bar{\varepsilon}) + o(||\varepsilon - \bar{\varepsilon}||)
\]

with the matrices

\[
\tilde{M} = \begin{pmatrix}
\nabla^2 \mathcal{L} & \nabla \bar{f}_1(\bar{x}, \bar{y}) & -\nabla g_1(\bar{x}, \bar{y}) & \cdots & -\nabla g_p(\bar{x}, \bar{y}) & \nabla \bar{f}_3(\bar{x}, \bar{y}) \\
-\bar{\mu}_1 \nabla \bar{f}_1(\bar{x}, \bar{y})^\top & \bar{\varepsilon}_1 - \bar{f}_1(\bar{x}, \bar{y}) & 0 & \cdots & 0 & 0 \\
\bar{\nu}_1 \nabla g_1(\bar{x}, \bar{y})^\top & 0 & g_1(\bar{x}, \bar{y}) & 0 & 0 & 0 \\
\vdots & 0 & 0 & \ddots & 0 & 0 \\
\bar{\nu}_p \nabla g_p(\bar{x}, \bar{y})^\top & 0 & 0 & 0 & g_p(\bar{x}, \bar{y}) & 0 \\
\nabla \bar{f}_3(\bar{x}, \bar{y}) & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

(with the gradients w. r. t. \((x, y)\)) and

\[
\tilde{N} = \left[ \begin{array}{c}
o_{2 \times (n_1 + 2)} \quad -\bar{\mu}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
0_{2 \times p} \quad -\bar{\mu}_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{array} \right]^\top.
\]

This sensitivity result gives us an approximately connection between the parameter \(\varepsilon\) and the minimal solutions \((x(\varepsilon), y(\varepsilon))\) of \((P_2(\varepsilon))\). As the minimal solutions \((x(\varepsilon), y(\varepsilon))\) are also approximation points of the set \(\mathcal{M}^0\) we can therefore control the location of additional points \((x(\varepsilon), y(\varepsilon))\) \(\in \mathcal{M}^0\) by choosing appropriate parameters \(\varepsilon\) using the following procedure.

We assume that the assumptions of Theorem 4.1 are satisfied. We vary the parameter \(\varepsilon\) by

\[
\begin{pmatrix}
\varepsilon_1 \\
\varepsilon_3
\end{pmatrix} = \begin{pmatrix}
\varepsilon_1 \\
\varepsilon_3
\end{pmatrix} + s^1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + s^2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

for \(s^1, s^2 \in \mathbb{R}\).

According to Theorem 4.1 we have the first-order approximation

\[
\begin{pmatrix}
x(\varepsilon) \\
y(\varepsilon)
\end{pmatrix} = \begin{pmatrix}
x(\bar{\varepsilon}) \\
y(\bar{\varepsilon})
\end{pmatrix} + \left( \tilde{M}^{-1} \tilde{N} \right)_{(x,y)} \begin{pmatrix}
s^1 \\
s^2
\end{pmatrix}
\]

with \((x(\bar{\varepsilon}), y(\bar{\varepsilon})) = (\bar{x}, \bar{y})\) and with \(\left( \tilde{M}^{-1} \tilde{N} \right)_{(x,y)}\) the first to the \(n_1 + 1\)-th rows of the matrix \(\tilde{M}^{-1} \tilde{N}\). Assuming \(s^2 = 0\) we approximately satisfy (9) for

\[
|s^1| = \beta \cdot \left( \left\| \left( \tilde{M}^{-1} \tilde{N} \right)_{(x,y)} \cdot v^1 \right\| \right)^{-1}
\]

(10)
and analogously for $s^1 = 0$ with

$$|s^2| = \beta \cdot \left( \left\| \left( \hat{M}^{-1} \hat{N} \right) |_{(x,y)} \cdot v^2 \right\| \right)^{-1}$$  (11)

According to the desired number $n^D \in \mathbb{N}$ of additional refinement points in each direction $y^i$, $i = 1, 2$, we set

$$\varepsilon = \bar{\varepsilon} + l^1 \cdot \begin{pmatrix} s^1 \\ 0 \end{pmatrix} + l^2 \cdot \begin{pmatrix} 0 \\ s^2 \end{pmatrix}, \quad l^1, l^2 \in \{-n^D, \ldots, n^D\} \subset \mathbb{Z}, \ (l^1, l^2) \neq (0, 0)$$

with $s^1$ as in (10) and $s^2$ as in (11). For these parameters $\varepsilon$ the problem $(P_2(\varepsilon))$ has to be solved. If there exists a minimal solution $(x(\varepsilon), y(\varepsilon))$ which also satisfies $(x(\varepsilon), y(\varepsilon)) \in \tilde{G}$ then an additional approximation point of the induced set $\Omega$ is found. Hence we can adaptively refine the induced set with a controlled distance between the refinement points. Of course such a refinement can be repeated arbitrarily often.

### 4.3 Algorithm

Finally we summarize the results of the preceding sections to an algorithm. Comments on the several steps are made afterwards.

**Step 1:** [Approximation of the induced set.]

Discretize the set $\tilde{G}^y$ by $\{y^1, \ldots, y^k\} \subset \tilde{G}^y$ and calculate for each $y^i, i = 1, \ldots, k$, a (nearly) equidistant approximation $\{x^{i,j}, \ldots, x^{i,n^i}\}$ ($n^i \in \mathbb{N}$) of the $K^1$-minimal solution set of the lower-level multiobjective optimization problem.

The set

$$A^0 := \{(x^{i,j}, y^i) \mid i = 1, \ldots, k, \ j = 1, \ldots, n^i\} \cap \tilde{G}$$

is an approximation of the induced set $\Omega$. Set $l := 0$.

**Step 2:** [Determination of minimal solutions.]

Determine the $K^2$-minimal points of the set $A^l$ w.r.t. the upper-level objective function, i.e. calculate

$$\mathcal{M}(F(A^l), K^2) = \{(\bar{x}, \bar{y}) \in A^l \mid \nexists (x, y) \in A^l \text{ such that } F(\bar{x}, \bar{y}) \in F(x, y) + K^2 \setminus \{0_2\}\}.$$ 

If this approximation of the minimal solution set of the original problem is satisfying stop, else continue with step 3.
Step 3: [Refinement of the induced set.]
Choose a refinement distance $\beta > 0$ and a number $n^D \in \mathbb{N}$ of refinement points (in each direction). Set $A^{l+1} := A^l$ and $l := l + 1$.

For all $(\bar{x}, \bar{y}) \in \mathcal{M}(F(A^{l-1}), K^2)$ repeat the following:

Let $(\bar{\mu}_1, \bar{\nu}, \bar{\mu}_3)$ be the Lagrange multipliers of the minimal solution $(\bar{x}, \bar{y})$ to the problem $(P_2(\bar{\varepsilon}))$ with $(\bar{\varepsilon}_1, \bar{\varepsilon}_3) = (f_1(\bar{x}, \bar{y}), \bar{y})$. Set

$$s_i := \beta \cdot \left(\| (\bar{M}^{-1} \bar{N})_{(x,y)} v_i \| \right)^{-1}, \quad i = 1, 2,$$

with $\bar{M}$, $\bar{N}$ as in Theorem 4.1 and $v^1 = (1, 0)^\top$, $v^2 = (0, 1)^\top$.

Solve $(P_2(\varepsilon))$ for all

$$\varepsilon = \bar{\varepsilon} + \left( \begin{array}{c} l_1^{l_1} s_1 \\ l_2^{l_2} s_2 \end{array} \right)$$

with $l_1, l_2 \in \{-n^D, \ldots, n^D\}$, $(l_1^{l_1}, l_2^{l_2}) \neq (0, 0)$.

If there exists a minimal solution $(x(\varepsilon), y(\varepsilon))$ of $(P_2(\varepsilon))$ and if it holds $(x(\varepsilon), y(\varepsilon)) \in \tilde{G}$ for this minimal solution, then set $A^l := A^l \cup \{(x(\varepsilon), y(\varepsilon))\}$.

Go to step 2.

In step 1 an appropriate procedure for determining high quality (in the sense of nearly equidistant) approximations of the $K^1$-minimal solution sets of biobjective optimization problems is needed. Such a solver is e.g. proposed in [15] based on the Pascoletti-Serafini scalarization. For solving these scalar problems a numerical solver should be applied which delivers also the Lagrange multipliers. As the $\varepsilon$-constraint problems can be seen as a special case of the Pascoletti-Serafini scalarization (see [14]) one has at the same time the Lagrange multipliers to the $\varepsilon$-constraint problems which are needed in step 3. If other methods for generating equidistant approximations are used in step 1 then the Lagrange multipliers needed in step 3 have to be calculated afterwards.

For the determination of the $K^2$-minimal solution set in step 2 a Pareto-filter as mentioned at the end of section 4.1 can be used. Applying the mentioned Jahn-Graef-Younes method can reduce the costs significantly. The resulting set is an approximation of the minimal solution set of the multiobjective bilevel optimization problem.

In step 3 several $\varepsilon$-constraint problems are solved. Thereby a numerical solver should be used which provides the Lagrange multipliers to the minimal solutions. These should also be stored for the case that in a later iteration a refinement around that minimal solution has to be done.

5 Numerical results

We now apply the described algorithm on two test problems. The first is just the problem of Example 2. Here the solution set is explicitly known and allows thus a comparison with the results gained with the algorithm presented in Section 4.3. The
second problem is based on test problem 1 presented in [15] extended with an upper level constraint being dependent on the lower-level variable. Thereby the former induced set is restricted. The advantage of both problems is that the induced set is a subset of $\mathbb{R}^3$ and thus visualizable.

5.1 Test problem 1

We consider the multiobjective optimization problem of Example 2 with the explicitly known induced set $\Omega$. We start with an approximation of the set $\Omega$ according to step 1 of the proposed algorithm. For this purpose we set $\tilde{G}^y = [0, 1]$ and discretize this set by $\{0.04, 0.12, 0.20, 0.28, \ldots, 0.96\}$. For the solution of the lower level biobjective optimization problems we use the solver presented in [15] with a predefined distance of $\alpha = 0.08$ between the approximation points. This results in the points shown as dots in Fig. 4.a). The points drawn in gray do not satisfy the condition $(x, y) \in \tilde{G}$ and hence only the black points approximate the induced set. The line shows the known solution set $S_{\min}$ of the bilevel problem, compare Example 2 and Fig. 3. In step 2 the set $\mathcal{M}(F(A^0), \mathbb{R}_+^2)$ is determined. These points are marked with circles in Fig. 4.a). The points shown in Fig. 4.a) are mapped under the upper-level objective function. The result is shown in Fig. 4,b). There, the efficient points, i.e. the image points of $\mathcal{M}(F(A^0), \mathbb{R}_+^2)$, are also connected with lines.

A refinement is done according to step 3 with $\beta = 0.7\alpha$. Thereby only around those points $(\tilde{x}, \tilde{y}) \in \mathcal{M}(F(A^0), \mathbb{R}_+^2)$ a refinement is done for which there exists no point $(x, y) \in \mathcal{M}(F(A^0), \mathbb{R}_+^2)$ with $\|F(\tilde{x}, \tilde{y}) - F(x, y)\| \leq 0.05$. The refinement points being an element of the set $\hat{G}$ are shown in Fig. 5 in gray.

A second refinement, again only around those points with no neighbor in the image space closer than 0.05, is done. Here we have chosen $\beta = 0.42\alpha$. The result is given in Fig. 6. One can see that the approximated efficient set and the actual, known efficient set of the bilevel problem are no longer visibly distinguishable. All refinements are done with $n^D = 2$. 

Figure 4: a) Approximation of the induced set and b) the image of this approximation under $F$. Non-dominated points are marked with circles.
Figure 5: a) Approximation of the induced set and b) the image of this approximation under $F$ after the first refinement.

Figure 6: a) Approximation of the induced set and b) the image of this approximation under $F$ after the second refinement.
5.2 Test problem 2

We extend the test problem presented in [15] by an additional constraint:

\[
\begin{align*}
\min_{x,y} & \left( \frac{F_1(x,y)}{F_2(x,y)} \right) = & \left( \frac{x_1 + x_2^2 + y + \sin^2(x_1 + y)}{\cos(x_2) \cdot (0.1 + y) \cdot \exp\left(-\frac{x_1}{0.1 + x_2}\right)} \right) \\
\text{subject to the constraints} & \\
x & \in \arg\min_{x} \left\{ \left( \frac{f_1(x,y)}{f_2(x,y)} \right) \mid (x,y) \in G \right\}, \\
(x_1 - 0.5)^2 + (x_2 - 5)^2 + (y - 5)^2 & \leq 16, \quad y \in [0,10]
\end{align*}
\]

with \( f_1, f_2 : \mathbb{R}^3 \to \mathbb{R} \),

\[
\begin{align*}
f_1(x,y) &= \frac{(x_1 - 2)^2 + (x_2 - 1)^2}{4} + x_2 y + \frac{(5 - y)^2}{16} + \sin\left(\frac{x_2}{10}\right), \\
f_2(x,y) &= \frac{x_1^2 + (x_2 - 6)^4 - 2x_1 y - (5 - y)^2}{80}
\end{align*}
\]

and

\[
G = \{(x_1, x_2, y) \in \mathbb{R}^3 \mid x_1^2 - x_2 \leq 0, \ 5x_1^2 + x_2 \leq 10, \ x_2 - (5 - y/6) \leq 0, \ x_1 \geq 0\}.
\]

Here it is \( n_1 = 2, \ n_2 = 1, \ m_1 = m_2 = 2, \ K^1 = K^2 = \mathbb{R}_+^2 \) and

\[
\tilde{G} = \{(x,y)^3 \mid (x_1 - 0.5)^2 + (x_2 - 5)^2 + (y - 5)^2 \leq 16, \ y \in [0,10]\}.
\]

For \( \tilde{G}^y \) we can choose \( \tilde{G}^y = [0,10] \).

We choose \( \alpha = 0.6 \) and solve the lower-level problems according to step 1 of the algorithm for \( y \in \{0,0.6,1.2,\ldots,9.6\} \). This first approximation is shown in Fig. 7,a). In black the generated points of the set \( \Omega \) are drawn and in gray the points which do not satisfy \((x, y) \in \tilde{G}\). The points which turn in step 2 out to be non-dominated are marked with circles. In Fig. 7,b) the image under the upper-level function \( F \) of the points in Fig. 7,a) is shown. The efficient points are circled and connected with a line.

Using the refinement strategy as in step 3, with \( n^D = 2 \) additional points in each direction and a refinement distance of \( \beta = 0.7\alpha \), leads (after selecting only those points which satisfy the constraint \((x, y) \in \tilde{G}\)) to the improved approximation of the induced set given in Fig. 8,a). In gray the refinement points are drawn. The image under the upper-level function is again given in Fig. 8,b). Finally a refinement with distance \( \beta = 0.49\alpha \) is done, but only for those non-dominated points where the image points have no neighbored point in the efficient set with a distance less than 0.2. The results are plotted in Fig. 9.

6 Conclusion

Solving bilevel optimization problems, especially without assuming convexity, is a difficult problem. Additional complexity is added allowing vector-valued objective
Figure 7: a) Approximation of the induced set and b) the image of this approximation under $F$. Non-dominated points are marked with circles.

Figure 8: a) Approximation of the induced set and b) the image of this approximation under $F$ after the first refinement.

Figure 9: a) Approximation of the induced set and b) the image of this approximation under $F$ after the second refinement.
functions. We have presented a numerical algorithm for the (approximated) solution of a multiobjective bilevel optimization problem with coupled upper-level constraints. We have also demonstrated the applicability of our procedure on two problems.

Some restrictions as on the dimension of the upper level variable and on the number of objectives on the lower level have been necessary. This is due to the fact that the efficient solution of multiobjective optimization problems with many objectives such that the solution set is well represented remains also a hard problem.

So far, the aim in multiobjective optimization is generally an approximation of the efficient set. Here, the usage of multiobjective optimization problems for solving the bilevel multiobjective optimization problems shows the importance of the development of methods which produce high quality approximations of the solution sets in the preimage space. Further these methods should be able to deal with arbitrary partial orderings or at least with orderings represented by a cone as $K = \mathbb{R}^m \times \{0_n\}$. Having such methods at hand the restrictions made here can be removed. This is subject to further research.

References


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